MODIFIED FIBERS AND LOCAL CONNECTEDNESS OF PLANAR CONTINUA

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Abstract. We describe non-locally connected planar continua via the concepts of modified fiber and numerical scale.

Given a compactum $X \subset \mathbb{C}$ and $x \in \partial X$, we show that the set of points $y \in \partial X$ that cannot be separated from $x$ by any finite set $C \subset \partial X$ is a continuum. This continuum is called the modified fiber $F^*_x$ of $X$ at $x$. If $x \in X^\circ$, we set $F^*_x = \{x\}$. When $F^*_x = \{x\}$ we show that the component of $X$ containing $x$ is locally connected at $x$. We also give an example of a planar continuum $X$, which is locally connected at a point $x \in X$ while the modified fiber $F^*_x$ is not trivial.

The modified scale $\ell^*(X)$ of non-local connectedness is then the least integer $p$ (or $\infty$ if such an integer does not exist) such that for each $x \in X$ there exist $k \leq p + 1$ subcontinua

$$X = N_0 \supset N_1 \supset N_2 \supset \cdots \supset N_k = \{x\}$$

such that $N_i$ is a modified fiber of $N_{i-1}$ for $1 \leq i \leq k$. If $X \subset \mathbb{C}$ is an unshielded continuum or a continuum whose complement has finitely many components, we obtain that local connectedness of $X$ is equivalent to the statement $\ell^*(X) = 0$.

We discuss the relation of our concepts to the works of Schleicher (1999) and Kiwi (2004). We further define an equivalence relation $\sim$ based on the modified fibers and show that the quotient space $X/\sim$ is a locally connected continuum. For connected Julia sets of polynomials and more generally for unshielded continua, we obtain that every prime end impression is contained in a modified fiber. Finally, we apply our results to examples from the literature and construct for each $n \geq 1$ examples of path connected continua $X_n$ with $\ell^*(X_n) = n$.

1. Introduction and main results

Motivated by the construction of Yoccoz puzzles used in the study on local connectedness of quadratic Julia sets and the Mandelbrot set, Schleicher [13] introduces a notion of fiber for full continua (continua $M \subset \mathbb{C}$ having a connected complement $\mathbb{C} \setminus M$), based on “separation lines” chosen from particular countable dense sets of external rays that land on points of $M$. Kiwi [7] uses finite “cutting sets” to define another version of fiber for Julia sets, even when they are not connected.

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Jolivet-Loridant-Luo [5] replace Schleicher’s “separation lines” with “good cuts”, i.e., simple closed curves \( J \) such that \( J \cap \partial M \) is finite and \( J \setminus M \neq \emptyset \).

In this way, Schleicher’s approach is generalized to continua \( M \subset \mathbb{C} \) whose complement \( \mathbb{C} \setminus M \) has finitely many components. For such a continuum \( M \), the pseudo-fiber \( E_x \) (of \( M \)) at a point \( x \in M \) is the collection of the points \( y \in M \) that cannot be separated from \( x \) by a good cut; the fiber \( F_x \) at \( x \) is the component of \( E_x \) containing \( x \). Here, a point \( y \) is separated from a point \( x \) by a simple closed curve \( J \) provided that \( x \) and \( y \) belong to different components of \( \mathbb{C} \setminus J \). And the point \( x \) may belong to the bounded or unbounded component of \( \mathbb{C} \setminus J \).

Clearly, the fiber \( F_x \) at \( x \) always contains \( x \). We say that a fiber or a pseudo-fiber is trivial if it coincides with the single point set \( \{ x \} \).

By [5, Proposition 3.6], every fiber of \( M \) is again a continuum with finitely many complementary components. Thus the hierarchy by “fibers of fibers” is well defined. This allows to define the scale \( \ell(M) \) of non-local connectedness as the least integer \( k \) such that for each \( x \in M \) there exist \( p \leq k + 1 \) subcontinua \( M = N_0 \supset N_1 \supset \cdots \supset N_p = \{ x \} \) such that \( N_i \) is a fiber of \( N_{i-1} \) for \( 1 \leq i \leq p \). If such an integer \( k \) does not exist, we set \( \ell(M) = \infty \).

In this paper, we rather follow Kiwi’s approach [7] and define “modified fibers” for compacta on the plane. The key point is: Kiwi focuses on Julia sets and uses “finite cutting sets” that consist of pre-periodic points, while we consider arbitrary compacta \( M \) on the plane (which may have interior points) and use “finite separating sets”. We refer to Example 7.1 for the difference between separating and cutting sets. Moreover, in Jolivet-Loridant-Luo [5], a good cut is not contained entirely in the underlying continuum \( M \). In the current paper, we will remove this assumption and only require that a good cut is a simple closed curve intersecting \( \partial M \) at finitely many points. After this slight modification we can establish the equivalence between the above mentioned two approaches, using good cuts or using finite separating sets. See Remark 1.3 for further details.

The notions and results will be presented in a way that focuses on the general topological aspects, rather than in the framework of complex analysis and dynamics.

**Definition 1.1.** Let \( X \subset \mathbb{C} \) be a (possibly disconnected) compact set. We will say that a point \( x \in \partial X \) is separated from \( y \in \partial X \) by a (possibly empty) subset \( C \subset X \) if there is a separation \( \partial X \setminus C = A \cup B \) with \( x \in A \) and \( y \in B \).
Here “$\partial X \setminus C = A \cup B$ is a separation” means that $\overline{A} \cap B = A \cap \overline{B} = \emptyset$ (or, equivalently, that $A$ and $B$ are relatively open in $\partial X \setminus C$).

- The modified fiber $F^*_x$ of $X$ at a point $x$ in the interior $X^o$ of $X$ is $\{x\}$; and the modified fiber $F^*_x$ of $X$ at a point $x \in \partial X$ is the set of the points $y \in \partial X$ that cannot be separated from $x$ by any finite set $C \subset X$. Clearly, every $F^*_x$ is closed in $X$; and the modified fiber of $X$ at any $x \in \partial X$ equals that of $\partial X$ at $x$. Actually, it is connected (Theorem 1).

- We inductively define a modified fiber of order $k \geq 2$ as a modified fiber of a continuum $Y \subset X$, where $Y$ is a modified fiber of order $k-1$.

- The local modified scale of non-local connectedness of $X$ at a point $x \in X$, denoted $\ell^*(X, x)$, is the least integer $p$ such that there exist $k \leq p + 1$ subcontinua

$$X = N_0 \supset N_1 \supset N_2 \supset \cdots \supset N_k = \{x\}$$

such that $N_i$ is a modified fiber of $N_{i-1}$ for $1 \leq i \leq k$. If such an integer does not exist we set $\ell^*(X, x) = \infty$.

- The (global) modified scale of non-local connectedness of $X$ is

$$\ell^*(X) = \sup\{\ell^*(X, x) : x \in X\}.$$  

We also call $\ell^*(X, x)$ the modified NLC-scale of $X$ at $x$, and $\ell^*(X)$ the modified NLC-scale of $X$.

We will firstly obtain the connectedness of $F^*_x$ and relate trivial modified fibers to local connectedness. Here, local connectedness at a particular point does not imply trivial modified fiber. In particular, let $\mathcal{K} \subset [0, 1]$ be Cantor’s ternary set, let $X$ be the union of $\mathcal{K} \times [0, 1]$ with $[0, 1] \times \{1\}$. See Figure 7. Then $X$ is locally connected at every $x = (t, 1)$ with $t \in \mathcal{K}$, while the modified fiber $F^*_x$ at this point is the whole segment $\{t\} \times [0, 1]$. See Example 7.5 for more details.

**Theorem 1.** Let $X \subset \mathbb{C}$ be a compact set. Then $F^*_x$ is connected for every $x \in X$. Moreover, $F^*_x = \{x\}$ implies that the component of $X$ containing $x$ is locally connected at $x$.

**Remark 1.2.** This theorem is related to Kiwi’s results [7, Corollaries 2.15 to 2.16] and can be considered as a generalization of these results to every compact set of the plane.

Secondly, we characterize modified fibers $F^*_x$ through simple closed curves $\gamma$ that separate $x$ from points $y$ in $X \setminus F^*_x$ and that intersect $\partial X$ at a finite
set or an empty set. This provides an equivalent way to develop the theory of modified fibers, for planar continua, and leads to a partial converse for the second part of Theorem 1. See Remark 1.3.

**Theorem 2.** Let \( x \in X \), where \( X \subset \mathbb{C} \) is a continuum. Then, \( F^*_x \) consists of all points of \( X \) which cannot be separated from \( x \) by a simple closed curve intersecting \( \partial X \) in a finite set.

This criterion is related to Kiwi’s characterization of fibers [7, Corollary 2.18], as will be explained at the end of Section 4.

**Remark 1.3.** We define a simple closed curve \( \gamma \) to be a **good cut of a continuum** \( X \subset \mathbb{C} \) if \( \gamma \cap \partial X \) is a finite set (the empty set is also allowed). We also say that two points \( x, y \in X \) are **separated by the good cut** \( \gamma \) if they lie in different components of \( \mathbb{C} \setminus \gamma \). This slightly weakens the requirements on “good cuts” in [5]. Therefore, given a continuum \( X \subset \mathbb{C} \) whose complement has finitely many components, the modified fiber \( F^*_x \) at any point \( x \in X \) is a subset of the fiber \( F_x \) at \( x \), if \( F_x \) is defined as in [5]. Consequently, we can infer that local connectedness of \( X \) implies triviality of all the modified fibers \( F^*_x \), by citing two of the four equivalent statements of [5, Theorem 2.2]: (1) \( X \) is locally connected; (2) every fiber \( F_x \) is trivial. Combining this with Theorem 1, we see that every continuum \( X \subset \mathbb{C} \) with \( \mathbb{C} \setminus X \) having finitely many components is locally connected if and only if every modified fiber \( F^*_x \) of \( X \) is trivial. The same result does not hold when the complement \( \mathbb{C} \setminus X \) has infinitely many components. Sierpinski’s universal curve gives a counterexample. However, for the connected Julia set \( J \) of a polynomial (whose complement may have infinitely many components), we may set \( X \) to be the **filled Julia set**, i.e. the union of \( J \) and all the bounded components of \( \mathbb{C} \setminus J \). Then the following holds.

(a) \( X \) is locally connected if and only if \( J \) is locally connected.

(b) The modified fiber of \( X \) at each \( z \in X^o \) is just the singleton \( \{z\} \) and the modified fiber of \( J \) at each \( x \in J = \partial X \) is equal to the modified fiber of \( X \) at \( x \).

The above item (a) is a direct corollary of [12, p.20, Theorem 2.1]. Therefore, the following statements also hold.

(c) Every modified fiber of \( X \) is trivial if and only if every modified fiber of \( J \) is trivial.

(d) \( J \) is locally connected if and only if \( F^*_x = \{x\} \) for each \( x \in J \).

**Remark 1.4.** The two approaches, via fibers \( F_x \) and via modified fibers \( F^*_x \), have their own merits. The former one follows Schleicher’s approach
and is more closely related to the theory of puzzles in the study of Julia sets and the Mandelbrot set; hence it may be used to analyse the structure of such continua by cultivating the dynamics of polynomials. The latter approach has a potential to be extended to the study of general compact metric spaces; and, at the same time, it is directly connected with the first approach when restricted to planar continua.

Thirdly, we define an equivalence relation \( \sim \) on \( X \) in Definition 1.5 and study the topology of \( X \) by investigating the quotient space \( X/\sim \). When \( X \) is connected, this quotient is a locally connected continuum. See Theorem 3 below. For this relation \( \sim \), every modified fiber \( F_\ast x \) is contained in a single equivalence class.

**Definition 1.5.** Let \( X \subset \mathbb{C} \) be a continuum. Let \( X_0 \) be the union of all the nontrivial modified fibers \( F_\ast x \) for \( x \in X \) and \( \overline{X_0} \) denote the closure of \( X_0 \). We define \( x \sim y \) if \( x = y \) or if \( x \neq y \) belong to the same component of \( \overline{X_0} \).

It is easy to see that \( \sim \) is an equivalence on \( X \) such that, for all \( x \in X \), the equivalence class \( [x]_\sim \) is a continuum containing the modified fiber \( F_\ast x \). Such a class equals \( \{x\} \) if only \( x \in (X \setminus \overline{X_0}) \) or \( \{x\} \) is a component of \( \overline{X_0} \). Since the components of \( \overline{X_0} \) form an upper semi-continuous decomposition of \( \overline{X_0} \), one may further verify that \( \{[x]_\sim : x \in X\} \) is also an upper semi-continuous decomposition. Hence \( \sim \) is a closed equivalence and that the natural projection \( \pi(x) = [x]_\sim \) is a monotone mapping, from \( X \) onto its quotient \( X/\sim \). For the details we refer to Lemma 5.1.

**Remark 1.6.** Actually, there is a more natural equivalence relation \( \approx \) by defining \( x \approx y \) whenever there exist points \( x_1 = x, x_2, \ldots, x_n = y \) in \( X \) such that \( x_i \in F_\ast x_{i-1} \). However, the relation \( \approx \) may not be closed, as a subset of the product \( X \times X \). On the other hand, if we take the closure of \( \approx \) we will obtain a closed relation, which is reflexive and symmetric but may not be transitive (see Example 7.3). The above Definition 1.5 solves this problem.

The following theorem provides important information about the topology of \( X/\sim \).

**Theorem 3.** Let \( X \subset \mathbb{C} \) be a continuum. Then \( X/\sim \) is metrizable and is a locally connected continuum, possibly a single point.

**Remark 1.7.** Kiwi [7] considers the special case where \( X \) is a component of the Julia set \( J(f) \) of a polynomial \( f \) with degree \( \geq 2 \) without irrationally neutral cycle. By Theorem 2, Corollary 1.1 and Theorem 3 of [7], the following interesting results concerning the structure of \( F_\ast x \) hold.
(a) If \( x \in J(f) \) is periodic or pre-periodic under \( f \) then \( F_x^* = \{ x \} \).

(b) If the modified fiber \( F_x^* \) of \( X \) at \( x \) is trivial then \( X \) is locally connected at \( x \).

(c) The modified fiber \( F_x^* \) contains the impression(s) of at least one and at most finitely many prime ends and the impression of any prime end is contained in \( F_x^* \) if only it intersects \( F_x^* \).

On the other hand, if \( X \) is the connected Julia set \( J \) of an arbitrary polynomial \( f \) the locally connected model introduced in [1] ensures that there is a finest monotone decomposition \( D \) of \( J \) such that the quotient space is a Peano continuum. In particular, every prime end impression is contained in a single element of \( D \). If \( f \) has no irrationally neutral cycle, from the results of [7] and [1] we can infer that every \( F_x^* \) lies in the single element of \( D \) that contains \( x \). Those results are of fundamental significance from the viewpoint of topology. They also play a crucial role in the study of complex dynamics. Actually, the restriction \( f|_J : J \to J \) induces a continuous map \( \tilde{f} : D \to D \) such that \( \pi \circ f = \tilde{f} \circ \pi \). Here \( \pi : J \to D \) is the natural projection sending \( x \in J \) to the unique element of \( D \) that contains \( x \). By Theorem 3, we also find some relations between modified fibers \( F_x^* \) and impressions of prime ends, when \( X \) is the Julia set of a polynomial (see Theorem 6.4). Combining this with laminations on the unit circle \( S^1 \subset \mathbb{C} \), the system \( f_\sim : J/\sim \to J/\sim \) is also a factor of the map \( z \mapsto z^n \) on \( S^1 \). However, it is not known yet whether the decomposition \( \{ [x]_\sim : x \in X \} \) in classes of \( \sim \) coincides with the finest decomposition \( D \) that corresponds to the locally connected model discussed in [1]. For more detailed discussions related to the dynamics of polynomials, we refer to [1, 7] and references therein.

Finally, to conclude the introduction, we propose the following problem.

**Problem 1.8.** To estimate the modified NLC-scale \( \ell^*(X) \) from above for particular continua \( X \subset \mathbb{C} \) such that \( \mathbb{C} \setminus X \) has finitely many components, and to compare the quotient space \( X/\sim \) with the locally connected model introduced in [1]. The Mandelbrot set or the Julia set of an infinitely renormalizable quadratic polynomial (when this Julia set is not locally connected) provide very typical choices for \( X \). In particular, the modified NLC-scale \( \ell^*(X) \) will be zero if the Mandelbrot set is locally connected, i.e., if \( MLC \) holds. In such a case, the relation \( \sim \) is trivial and its quotient is immediate.
Remark 1.9. Section 7 gives several examples of continua $X \subset \mathbb{C}$. For those continua $X$, we obtain the decomposition $\{[x]_\sim : x \in X\}$ into sub-continua and represent the quotient space $X/\sim$ on the plane. For those examples, the modified NLC-scale $\ell^*(X)$ is easy to determine.

Remark 1.10. The equivalence classes $[x]_\sim$ mentioned in Theorem 3 form a concrete upper semi-continuous decomposition of an arbitrary continuum $X$ on the plane, with the property that the quotient space $X/\sim$ is a locally connected continuum. In the special case $X$ is unshielded, i.e., $X$ is equal to the boundary of the unbounded component of $\mathbb{C} \setminus X$, the finest decomposition in [1, Theorem 1] is finer than or equal to our decomposition $\{[x]_\sim : x \in X\}$. We refer to Theorem 6.4 for details when $X$ is assumed to be unshielded. It is not known whether those two decompositions actually coincide. If the answer is yes, the quotient space $X/\sim$ in Theorem 3 is exactly the finest locally connected model of $X$, which shall be in some sense “computable”. Here, an application of some interest is to study the locally connected model of an infinitely renormalizable Julia set [4] or of the Mandelbrot set, as mentioned in Problem 1.8.

We arrange our paper as follows. Section 2 recalls some basic notions and results from topology that are closely related to local connectedness. Sections 3, 4 and 5 respectively prove Theorems 1, 2 and 3. Section 6 discusses basic properties of modified fibers, studies those continua from a viewpoint of dynamic topology (as proposed by Whyburn and Duda [16, pp.130-144]) and relates the theory of modified fibers to the theory of prime ends for unshielded continua. Finally, in Section 7, we illustrate our results through examples from the literature and give an explicit sequence of path connected continua $X_n$ satisfying $\ell^*(X_n) = n$.

2. A Revisit to Local Connectedness

Definition 2.1. A topological space $X$ is locally connected at a point $x_0 \in X$ if for any neighborhood $U$ of $x_0$ there exists a connected neighborhood $V$ of $x_0$ such that $V \subset U$, or equivalently, if the component of $U$ containing $x_0$ is also a neighborhood of $x_0$. The space $X$ is then called locally connected if it is locally connected at every of its points.

We focus on metric spaces and their subspaces. The following characterization can be found as the definition of locally connectedness in [16, Part A, Section XIV].
Lemma 2.2. A metric space \((X,d)\) is locally connected at \(x_0 \in X\) if and only if for any \(\varepsilon > 0\) there exists \(\delta > 0\) such that any point \(y \in X\) with \(d(x_0,y) < \delta\) is contained together with \(x_0\) in a connected subset of \(X\) which is of diameter less than \(\varepsilon\).

When \(X\) is compact, Lemma 2.2 is a local version of [9, p.183, Lemma 17.13(d)]. For the convenience of the readers, we give here the concrete statement as a lemma.

Lemma 2.3. A compact metric space \(X\) is locally connected if and only if for every \(\varepsilon > 0\) there is \(\delta > 0\) so that any two points of distance less than \(\delta\) are contained in a connected subset of \(X\) which is of diameter less than \(\varepsilon\).

Using Lemma 2.2, we obtain a fact concerning continua of the Euclidean space \(\mathbb{R}^n\).

Lemma 2.4. Let \(X \subset \mathbb{R}^n\) be a continuum and \(U = \bigcup_{\alpha \in I} W_\alpha\) the union of any collection \(\{W_\alpha : \alpha \in I\}\) of components of \(\mathbb{R}^n \setminus X\). If \(X\) is locally connected at \(x_0 \in X\), then so is \(X \cup U\). Consequently, if \(X\) is locally connected, then so is \(X \cup U\).

Proof. Choose \(\delta\) with properties from Lemma 2.2 with respect to \(x_0, X\) and \(\varepsilon/2\). For any \(y \in U\) with \(d(x_0,y) < \delta\) we consider the segment \([x_0,y]\) between \(x_0\) and \(y\). If \([x_0,y] \subset (X \cup U)\), we are done. If not, choose the point \(z \in ([x_0,y] \cap X)\) that is closest to \(y\). Since \(y \in U\) lies in a component \(W_\alpha\) of \(\mathbb{R}^n \setminus X\) and since \(W_\alpha \subset U\), the segment \([y,z]\) is contained in \(\overline{W_\alpha}\) and hence is a subset of \(X \cup U\). By the choice of \(\delta\) and Lemma 2.2, we may connect \(z\) and \(x_0\) with a continuum \(A \subset X\) of diameter less than \(\varepsilon/2\). Therefore, the continuum \(B := A \cup [y,z] \subset (X \cup U)\) is of diameter at most \(\varepsilon\) as desired. \(\square\)

In the present paper, we are mostly interested in continua on the plane, especially continua \(X\) which are on the boundary of a continuum \(M \subset \mathbb{C}\). Typical choice of such a continuum \(M\) is the connected filled Julia set of a rational function. Several results from [16] will be very helpful in our study.

The first result gives a fundamental fact about a continuum failing to be locally connected at one of its points. The proof can be found in [16, p.124, Corollary].

Lemma 2.5. If a continuum \(M\) is not locally connected at a point \(p\) then it is not locally connected at all points of a nondegenerate subcontinuum of \(M\) that contains \(p\).

The second result will be referred to as Torhorst Theorem (see [16, p.124, Torhorst Theorem] and [16, p.126, Lemma 2]).
Lemma 2.6. The boundary $B$ of each component $C$ of the complement of a locally connected continuum $M$ is itself a locally connected continuum. If further $M$ has no cut point, then $B$ is a simple closed curve.

We finally recall a Plane Separation Theorem [16, p.120, Exercise 2].

Proposition 2.7. If $A$ is a continuum and $B$ is a closed connected set of the plane with $A \cap B$ being a totally disconnected set, and with $A \setminus B$ and $B \setminus A$ being connected, then there exists a simple closed curve $J$ separating $A \setminus B$ and $B \setminus A$ such that $J \cap (A \cup B) \subset (A \cap B)$.

3. Fundamental properties of modified fibers

The proof for Theorem 1 has two parts. We start from the connectedness of modified fibers $F^*_x$. Since $F^*_x = \{x\}$ for all $x \in X$, we only consider the modified fibers $F^*_x$ for $x \in \partial X$, which just equals the modified fiber of $\partial X$ at $x$.

Theorem 3.1. Let $X \subset \mathbb{C}$ be a compact set. Then every $F^*_x$ is connected.

Proof. Suppose on the contrary that $F^*_x$ is disconnected for some $x \in \partial X$. Then we can fix a separation $F^*_x = A \cup B$ with $x \in A$ and $B \neq \emptyset$. Fix a point $x' \in B$. Because $A$ and $B$ are compact and disjoint nonempty sets, they have some open neighbourhoods $A^*$ (of $A$) and $B^*$ (of $B$) with disjoint closures. Then $K = \partial X \setminus (A^* \cup B^*)$ is a compact subset of $\partial X$. As $F^*_x \cap K = \emptyset$, we may find for each $z \in K$ a finite set $C_z$ and a separation $\partial X \setminus C_z = U_z \cup V_z$ such that $x \in U_z$, $z \in V_z$ and both of $U_z$ and $V_z$ are relatively open in $\partial X$. By flexibility of $z \in K$, we obtain an open cover $\{V_z : z \in K\}$ of $K$, which then has a finite subcover $\{V_{z_1}, \ldots, V_{z_n}\}$. Let

$$U = U_{z_1} \cap \cdots \cap U_{z_n}, \quad V = V_{z_1} \cup \cdots \cup V_{z_n}.$$ 

Then $U, V$ are disjoint sets open in $\partial X$ such that $C := \partial X \setminus (U \cup V)$ is a subset of $C_{z_1} \cup \cdots \cup C_{z_n}$, hence it is also a finite set. Let $C' = C \setminus \{x'\}$. Then:

$$\partial X \setminus C' = U \cup \{x'\} \cup V = (U \cap A^*) \cup ((U \cap B^*) \cup \{x'\} \cup V)$$

and the right-hand side is a separation with $x \in U \cap A^*$ and $x' \in (U \cap B^*) \cup \{x'\} \cup V$. (This separation follows from the fact that each of the sets $(U \cap B^*) \cup \{x'\}$ and $V$ is separated from $U \cap A^*$, since $U$ is separated from $V$ and $A^*$ from $B^*$, and hence from $\{x'\}$). This contradicts the assumption
that \( x' \in F^*_x \), because \( F^*_x \) being the modified fiber at \( x \), none of its points can be separated from \( x \) by the finite set \( C' \).

Then we recover in fuller generality that triviality of the modified fiber at a point \( x \) in a continuum \( X \subset \mathbb{C} \) implies local connectedness of \( X \) at \( x \). More restricted versions of this result appear earlier: in [13] for continua in the plane with connected complement, in [7] for Julia sets of monic polynomials or the components of such a set, and in [5] for continua in the plane whose complement has finitely many components. In the remaining part of this section, we denote by \( B(x, r) \) the open disk centered at \( x \) with radius \( r > 0 \).

**Theorem 3.2.** If \( F^*_x = \{x\} \) for a point \( x \) in a continuum \( X \subset \mathbb{C} \) then \( X \) is locally connected at \( x \).

**Proof.** We only consider the case \( x \in \partial X \) and prove that if \( X \) is not locally connected at \( x \) then \( F^*_x \) contains a non-degenerate continuum \( M' \subset \partial X \).

By definition, if \( X \) is not locally connected at \( x \) there exists a number \( r > 0 \) such that the component \( Q_x \) of \( B(x, r) \cap X \) containing \( x \) is not a neighborhood of \( x \) in \( X \). This means that there exists a sequence of points \( \{x_k\}_{k=1}^\infty \subset (B(x, r) \cap X) \setminus Q_x \) such that \( \lim_{k \to \infty} x_k = x \). Let \( Q_k \) be the component of \( B(x, r) \cap X \) containing \( x_k \). By [11, p.74, Boundary Bumping Theorem II] applied to \( X \) and its proper subset \( B(x, r) \cap X \), it follows that \( Q_x \) as well as all the sets \( Q_k \) intersect \( \partial B(x, r) \). Moreover, \( Q_i \cap \{x_k\}_{k=1}^\infty \) is a finite set for each \( i \geq 1 \); hence we may assume, by taking a subsequence, that \( Q_i \cap Q_j = \emptyset \) for \( i \neq j \).

Since the nonempty compact subsets of \( X \) form a compact metric space under Hausdorff distance, we may further assume that there exists a continuum \( M \) such that \( \lim_{k \to \infty} Q_k = M \) under Hausdorff distance. Clearly, we have \( x \in M \subset Q_x \). Note that \( Q_x \), and thus \( M \), may not be contained in \( \partial X \). In the sequel, we will obtain a non-degenerate sub-continuum \( M' \subset M \) with \( x \in M' \subset \partial X \).
Let $W_k$ be the unbounded component of $C \setminus Q_k$. Then $x \in W_k$ for all $k \geq 1$ and every $\partial W_k$ is a continuum intersecting $\partial B(x, r)$. Note that it is possible that $\partial B(x, r) \cap \partial W_k \cap X^o \neq \emptyset$. Hence, let us set $E_k := B(x, r) \cap \partial W_k$. Then $E_k \subset \partial X$. Indeed, by definition of $W_k$, we have $\partial W_k \subset \partial Q_k \subset Q_k$. Therefore, if $y \in E_k \cap X^o$, then $y \in B(x, r) \cap X^o \cap Q_k$, hence $y \in Q_k^o$, a contradiction to $y \in \partial Q_k$.

Then, we consider the segment $xx_k$ and denote by $y_k$ the last point (from $x_k \in Q_k$ to $x$) of this segment that also lies on $Q_k$. Note that $y_k \in E_k$ for all $k \geq 1$ and that $\lim_{k \to \infty} y_k = \lim_{k \to \infty} x_k = x$.

Applying [11, p.74, Boundary Bumping Theorem II] to $\partial W_k$ and its proper subset $E_k$, we see that the closure of the component $Q'_k$ of $E_k$ containing $y_k$ intersects $\partial B(x, r)$. Since $E_k \subset \partial X$, we have $\overline{Q'_k} \subset \partial X$ and $\overline{Q'_k} \cap \partial B(x, r) \neq \emptyset$. Therefore, we may fix an appropriate sub-sequence of $\{Q'_k\}$ converging to a limit continuum $M'$ under Hausdorff distance.

Clearly, every continuum $\overline{Q'_k}$ intersects $\partial B(x, r)$, so does the limit $M'$. On the other hand, as $y_k \in Q'_k \subset \partial X$ and $\lim_{k \to \infty} y_k = x$, we also have $x \in M' \subset \partial X$. In particular, $M'$ is non-degenerate. Moreover, no point $y \in M' \setminus \{x\}$ can be separated from $x$ in $\partial X$ by a finite set $C$, since $\partial X \setminus C$ includes $(M' \cup \bigcup_{k \geq 1} Q'_k) \setminus C$ as a subset, which contains a sub-sequence of the above continua $\overline{Q'_k}$ converging to $M' \supset \{x, y\}$. This indicates that $F_x^*$ contains the non-degenerate continuum $M'$ thus is not trivial. \[\square\]

Note that Theorem 3.1 proves the first statement of Theorem 1. For the second statement of Theorem 1, assuming that the modified fiber of $X$ at $x$ is trivial implies that the modified fiber of the component $Q_x$ of $X$ containing $x$ is also trivial. We can then apply Theorem 3.2 to $Q_x$.

4. SCHLEICHER’S AND KIWI’S APPROACHES UNIFIED

The following proposition implies Theorem 2. We present it in this form, since it can be seen as a modification of the plane separation theorem (Proposition 2.7).

**Proposition 4.1.** Let $C$ be a finite subset of a continuum $X \subset \mathbb{C}$ and $x, y$ two points on $X \setminus C$. If there is a separation $X \setminus C = P \cup Q$ with $x \in P$ and $y \in Q$ then $x$ is separated from $y$ by a simple closed curve $\gamma$ with $(\gamma \cap X) \subset C$.

**Proof.** By [11, p.73, Boundary Bumping Theorem I], we know that every component of $\overline{P}$ intersects $C$. Now, since $C$ is a finite set, it follows that $\overline{P}$ has finitely many components, say $P_1, \ldots, P_k$. We may assume that $x \in P_1$. 

Similarly, every component of \( \overline{Q} \subset (Q \cup C) \) intersects \( C \) and \( \overline{Q} \) has finitely many components, say \( Q_1, \ldots, Q_l \). We may assume that \( y \in Q_1 \).

Let \( P_1^* = P_2 \cup \cdots \cup P_k \cup Q_1 \cup \cdots \cup Q_l \). Then \( X = P_1 \cup P_1^* \), \( x \in P_1 \), \( y \in P_1^* \) and \( (P_1 \cap P_1^*) \subset C \). Let \( N_1 = \{ z \in P_1 : \text{dist}(z, P_1^*) \geq 1 \} \) and for each \( j \geq 2 \), let

\[
N_j = \{ z \in P_1 : 3^{-j} \leq \text{dist}(z, P_1^*) \leq 3^{-j+1} \}.
\]

Clearly, every \( N_j \) is a compact set. Therefore, we may cover \( N_j \) by finitely many open disks centered at a point in \( N_j \) and with radius \( r_j = 3^{-j-1} \), say \( B(x_{j1}, r_j), \ldots, B(x_{jk(j)}, r_j) \).

For \( j \geq 1 \), let us set \( M_j = \bigcup_{i=1}^{k(j)} B(x_{ji}, r_j) \). Then \( M = \bigcup_{j \geq 1} M_j \) is a compact set containing \( P_1 \). Its interior \( M^o \) contains \( x \). Moreover, \( P_1^* \cap \bigcup_{j \geq 1} M_j = \emptyset \) by definition of \( N_j \) and \( M_j \), while \( M \setminus \bigcup_{j \geq 1} M_j \) is a subset of \( P_1 \cap P_1^* \), hence we have \( M \cap P_1^* = P_1 \cap P_1^* \) and \( y \notin M \). Also, \( \partial M \cap X \) is a subset of \( P_1 \cap P_1^* \), hence it is a finite set.

Now \( M \) is a continuum, since \( P_1 \) is itself a continuum and the disks \( B(x_{ji}, r_j) \) are centered at \( x_{ji} \in N_j \). The continuum \( M \) is even locally connected at every point on \( M \setminus C = \bigcup_j M_j \). Indeed, it is locally a finite union of disks, since \( M_j \cap M_k = \emptyset \) as soon as \(|j - k| > 1\) and since every point of \( M \setminus C \) is in one of these disks. As \( C \) is finite, it follows from Lemma 2.5 that \( M \) is a locally connected continuum.

Now, let \( U \) be the component of \( \mathbb{C} \setminus M \) that contains \( y \). By Torhorst Theorem, see Lemma 2.6, the boundary \( \partial U \) of \( U \) is a locally connected continuum. Therefore, by Lemma 2.4, the union \( U \cup \partial U \) is also a locally connected continuum. Since \( U \) is a complementary component of \( \partial U \), the union \( U \cup \partial U \) even has no cut point. It follows from Torhorst Theorem that the boundary \( \partial V \) of any component \( V \) of \( \mathbb{C} \setminus (U \cup \partial U) \) is a simple closed curve. Note that this curve separates every point of \( U \) from any point of \( V \).

Choosing \( V \) to be the component of \( \mathbb{C} \setminus (U \cup \partial U) \) containing \( x \), we obtain a simple closed curve \( J = \partial V \) separating \( y \) from \( x \).

Finally, since \( J = \partial V \subset \partial U \subset \partial M \), we see that \( J \cap X \) is contained in the finite set \( C \). Consequently, \( J \) is a good cut of \( X \) separating \( x \) from \( y \).

This result proves Theorem 2 and is related to Kiwi’s characterization of fibers. Restricting to connected Julia sets \( J(f) \) of polynomials \( f \), Kiwi [7] defines for \( \zeta \in J(f) \) the fiber \( \text{Fiber}(\zeta) \) as the set of \( \xi \in J(f) \) such that \( \xi \) and \( \zeta \) lie in the same connected component of \( J(f) \setminus Z \) for every finite set \( Z \subset J(f) \), made up of periodic or preperiodic points that are not in the grand orbit of a Cremer point. Here we want to note that \( J(f) \setminus Z \) always

---

\(^1\)This idea is inspired by the proof of the plane separation theorem, see Proposition 2.7.
has finitely many components. Kiwi showed in [7, Corollary 2.18] that these fibers Fiber(ζ) can be characterized by using separating curves involving external rays.

5. A locally connected model for the continuum X

This section recalls a few notions and results from Kelley’s General Topology [6] and proves Theorem 3 in two steps:

(1) $X/\sim$ is metrizable, hence is a compact connected metric space, i.e., a continuum.

(2) $X/\sim$ is a locally connected continuum.

A decomposition $\mathcal{D}$ of a topological space $X$ is upper semi-continuous if for each $D \in \mathcal{D}$ and each open set $U$ containing $D$ there is an open set $V$ such that $D \subset V \subset U$ and $V$ is the union of members of $\mathcal{D}$ [6, p.99]. Given a decomposition $\mathcal{D}$, we may define a projection $\pi : X \to \mathcal{D}$ by setting $\pi(x)$ to be the unique member of $\mathcal{D}$ that contains $x$. Then, the quotient space $\mathcal{D}$ is equipped with the largest topology such that $\pi : X \to \mathcal{D}$ is continuous.

By [11, p.40, Theorem 3.9], any upper semi-continuous decomposition of a compact metric space is metrizable. Therefore, the following lemma ensures that $X/\sim$ is compact and metrizable.

Lemma 5.1. The decomposition $\{[x]_\sim : x \in X\}$ is upper semi-continuous.

Proof. Let $X_0$ be the union of all the nontrivial $F_x^*$. In particular, $\overline{X_0} \subset \partial X$. Clearly, the above decomposition $\{[z]_\sim : x \in X\}$ consists of all components of the closed subset $\overline{X_0}$ of the compact space $X$, and of all singletons in $X \setminus \overline{X_0}$. Hence the assertion follows from the well known fact that such a decomposition is upper semi-continuous. □

Theorem 5.2. The quotient $X/\sim$ is a locally connected continuum.

Proof. As $X$ is a continuum, $\pi(X) = X/\sim$ is itself a continuum. We now prove that this quotient is locally connected. If $V$ is an open set in $X/\sim$ that contains $[x_0]_\sim$, as an element of $X/\sim$, then the pre-image $U := \pi^{-1}(V)$ is open in $X$ and contains the class $[x_0]_\sim$ as a subset. We shall prove that $V$ contains a connected neighborhood of $[x_0]_\sim$.

We assume $U \neq X$ and let $Q$ be the component of $U$ containing $[x_0]_\sim$. Then, applying [11, p.74, Boundary Bumping Theorem II] to $U \subset X$ we have $\overline{Q} \setminus U \neq \emptyset$. Note that we only need to verify that $Q$ is also a neighborhood of $[x_0]_\sim$, since the monotonicity of the projection $\pi(z) = [z]_\sim$ will then ensure that $Q$ contains a saturated neighborhood of $x_0$, say $W$, whose image under $\pi$ is a neighborhood of the point $[x_0]_\sim$ in $V$. 
Suppose on the contrary that $Q$ is not a neighborhood of $[x_0]_{\sim}$. Then we can find an infinite sequence $\{x_k\}$ in $U \setminus Q$ with $\lim_{k \to \infty} x_k = x \in [x_0]_{\sim}$. Here the component $Q_k$ of $U$ containing $x_k$ satisfies $Q_k \cap Q = \emptyset$. And we may assume that $Q_i \cap Q_j = \emptyset$ for $i \neq j$. Applying [11, p.74, Boundary Bumping Theorem II] to the continuum $X$ and its proper open subset $U$, we have $\overline{Q_k} \setminus U = \overline{Q_k} \cap \partial U \neq \emptyset$.

Cover the compact set $X \setminus U$ by finitely many open disks $D_1, \ldots, D_n$ with centers $y_i \in (X \setminus U)$ and radius $r < \frac{1}{3}r_0$, where $r_0$ is the distance between the closed sets $X \setminus U$ and $[x_0]_{\sim}$. Those disks $D_i$ may be chosen so that $|y_i - y_j| \neq 2r$ for all $i \neq j$. Then $Y = \bigcup_i D_i$ is disjoint from $[x_0]_{\sim}$ and every of its components is a continuum having no cut point.

By Torhorst Theorem, cited as Lemma 2.6 in Section 2 of this paper, the boundary $\partial Y$ consists of finitely many Jordan curves. The choice of the disks $\{D_i\}$ implies that those Jordan curves are pairwise disjoint. Hence the component $U_Y$ of $C \setminus Y$ containing $x$ is topologically equivalent to a circle domain, i.e., the difference between the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and finitely many pairwise disjoint closed disks contained in $\mathbb{D}$.

By the choices of the disks $D_1, \ldots, D_n$ we can infer that $Y \cap [x_0]_{\sim} = \emptyset$. Thus we have $[x_0]_{\sim} \subset U_Y$. By truncating finitely many $x_i$ we may assume that $x_k \in U_Y$ for all $k \geq 1$. Since the continua $\overline{Q_1}, \overline{Q_2}, \ldots$, all intersect $\partial U \subset Y \subset (C \setminus U_Y)$, there is a component $\Gamma$ of $\partial U_Y$ such that $\overline{Q_k} \cap \Gamma \neq \emptyset$ for infinitely many (and, w.l.o.g., for all) $k \geq 1$. This $\Gamma$ is a Jordan curve, because $U_Y$ is a circle domain.

By Schönflies Theorem [10, pp.71-72, Theorem 3 and 4], we may fix a homeomorphism $\varphi : \mathbb{C} \to \mathbb{C}$ sending the component of $\mathbb{C} \setminus \Gamma$ that contains $x$ (and hence $[x_0]_{\sim}$) onto the open unit disk $\mathbb{D}$. For all $k \geq 1$ let $Q'_k$ be the component of $\overline{Q_k} \cap U_Y$ containing $x_k$. Now applying [11, p.74, Boundary Bumping Theorem II] to the continuum $\overline{Q_k}$ and its proper open subset $\overline{Q_k} \cap U_Y$, we know that $\overline{Q_k} \cap \Gamma \neq \emptyset$. For the same reasons, the component $Q'$ of $\overline{Q} \cap U_Y$ containing $x$ satisfies $\overline{Q'} \cap \Gamma \neq \emptyset$.

Let $W_k$ be the unbounded component of $\mathbb{C} \setminus \varphi(\overline{Q_k})$. Then $\partial W_k$ is a continuum intersecting $\partial \mathbb{D} = \varphi(\Gamma)$ such that $E_k = \mathbb{D} \cap \partial W_k \subset \varphi(\partial X)$. Here it is possible that $\partial \mathbb{D} \cap \partial W_k \cap \varphi(X') \neq \emptyset$. Moreover, we have $\varphi(x) \in W_k$ for all $k \geq 1$, since $\varphi(\overline{Q'})$ can be connected to infinity by an external ray of $\mathbb{D}$ landing on a point in $\varphi(\overline{Q'}) \cap \partial \mathbb{D}$.

Let $y_k$ be the last point lying on $\varphi(\overline{Q_k})$ from $\varphi(x_k)$ to $\varphi(x)$ along the segment $\varphi(x_k)\varphi(x) \subset \mathbb{D}$. Then $y_k \in E_k$ for all $k \geq 1$; moreover, we have $\lim_{k \to \infty} \varphi^{-1}(y_k) = x$. Let $Q''_k$ be the component of $E_k$ containing $y_k$. Then
Boundary Bumping Theorem II [11, p.74] ensures that $\overline{Q'_k}$ is a continuum that intersects $\partial \mathbb{D}$. Thus we have $\varphi^{-1}(\overline{Q'_k}) \cap \Gamma \neq \emptyset$ for all $k \geq 1$.

Finally, by the containments $\varphi^{-1}(\overline{Q'_k}) \subset \varphi^{-1}(E_k) \subset X$, we may choose a subsequence $M_i = \varphi^{-1}(Q'_k(i))$ that converges to a limit continuum $M'$ under Hausdorff distance. Then we have $x \in M' \subset [x_0]_\sim$ and $M' \cap \Gamma \neq \emptyset$. This is absurd, since we have $[x_0]_\sim \subset U_Y$ and $\Gamma \subset \partial U_Y$. □

6. How modified fibers are changed under continuous maps

This section discusses how modified fibers are changed under continuous maps. As a special application, we may compare the dynamics of a polynomial $f_c(z) = z^n + c$ on its Julia set $J_c$, the expansion $z \mapsto z^d$ on unit circle, and an induced map $\tilde{f}_c$ on the quotient $J_c/\sim$.

Let $X, Y \subset \mathbb{C}$ be continua and $x \in X$ a point. The first primary observation is that $f(F^*_x) \subset F^*_x(x)$ for any finite-to-one continuous surjection $f : X \to Y$.

Indeed, for any $y \neq x$ in the modified fiber $F^*_x$ and any finite set $C \subset Y$ that is disjoint from $\{f(x), f(y)\}$, we can see that $f^{-1}(C)$ is a finite set disjoint from $\{x, y\}$. Since $y \in F^*_x$ there exists no separation $X \setminus f^{-1}(C) = A \cup B$ with $x \in A, y \in B$; therefore, there exists no separation $Y \setminus C = P \cup Q$ with $f(x) \in P, f(y) \in Q$. This certifies that $f(y) \in F^*_x(x)$.

The inclusion $f(F^*_x) \subset F^*_x(x)$ indicates that $f(X_0) \subset Y_0$, where $X_0$ is the union of all the nontrivial modified fibers $F^*_x$ in $X$, and $Y_0$ the union of those in $Y$. It follows that $f([x]_{\sim}) \subset [f(x)]_{\sim}$. Therefore, the correspondence $[x]_{\sim} \mapsto [f(x)]_{\sim}$ gives a well defined map $\tilde{f} : X/\sim \to Y/\sim$ that satisfies the following commutative diagram, in which each downward arrow $\downarrow$ indicates the natural projection from a space onto its quotient.

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
X/\sim & \xrightarrow{\tilde{f}} & Y/\sim
\end{array}
$$

Given an open set $U \subset Y/\sim$, we can use the definition of quotient topology to infer that $V := \tilde{f}^{-1}(U)$ is open in $X/\sim$ whenever $\pi^{-1}(V)$ is open in $X$. On the other hand, the above diagram ensures that $\pi^{-1}(V) = f^{-1}(\pi^{-1}(U))$, which is an open set of $X$, by continuity of $f$ and $\pi$.

The above arguments lead us to a useful result for the study of dynamics on Julia sets.
Theorem 6.1. Let \( X, Y \subset \mathbb{C} \) be continua. Let the relation \( \sim \) be defined as in Theorem 3. If \( f : X \to Y \) is continuous, surjective and finite-to-one then \( \tilde{f}(\{x\}_\sim) := [f(x)]_\sim \) defines a continuous map with \( \pi \circ f = \tilde{f} \circ \pi \).

Remark 6.2. Every polynomial \( f_c(z) \) restricted to its Julia set \( J_c \) satisfies the conditions of Theorem 6.1, if we assume that \( J_c \) is connected; so the restricted system \( f_c : J_c \to J_c \) has a factor system \( \tilde{f}_c : J_c / \sim \to J_c / \sim \), whose underlying space is a locally connected continuum.

Let \( X \subset \mathbb{C} \) be an unshielded continuum. Let \( U_\infty \) be the unbounded component of \( \mathbb{C} \setminus X \). Here, \( X \) is unshielded provided that \( X = \partial U_\infty \). Let \( \mathbb{D} := \{ z \in \hat{\mathbb{C}} : |z| < 1 \} \) be the unit disk. By Riemann Mapping Theorem, there exists a conformal isomorphism \( \Phi : \hat{\mathbb{C}} \setminus \mathbb{D} \to U_\infty \) that fixes \( \infty \) and has positive derivative at \( \infty \). The prime end theory \([2, 15]\) builds a correspondence between an angle \( \theta \in S^1 := \partial \mathbb{D} \) and a continuum \( \text{Imp}(\theta) := \{ w \in X : \exists z_n \in \mathbb{D} \text{ with } z_n \to e^{i\theta}, \lim_{n \to \infty} \Phi(z_n) = w \} \). We call \( \text{Imp}(\theta) \) the impression of \( \theta \). By \([3, \text{p.173, Theorem 9.4}]\), we may fix a simple open arc \( R_\theta \) in \( \mathbb{C} \setminus \mathbb{D} \) landing at the point \( e^{i\theta} \) such that \( \Phi(R_\theta) \cap X = \text{Imp}(\theta) \).

We will connect impressions to modified fibers. Before that, we obtain a useful lemma concerning good cuts of an unshielded continuum \( X \) on the plane. Here a good cut of \( X \) is a simple closed curve that intersects \( X \) at a finite subset (see Remark 1.3).

Lemma 6.3. Let \( X \subset \mathbb{C} \) be an unshielded continuum and \( U_\infty \) the unbounded component of \( \mathbb{C} \setminus X \). Let \( x \) and \( y \) be two points on \( X \) separated by a good cut \( \gamma \) of \( X \). Then we can find a good cut separating \( x \) from \( y \) that intersects \( U_\infty \) at an open arc.

Proof. Since each of the two components of \( \mathbb{C} \setminus \gamma \) intersects \( \{x, y\} \), we have \( \gamma \cap U_\infty \neq \emptyset \). Since \( \gamma \cap X \) is a finite set, the difference \( \gamma \setminus X \) has finitely many components. Let \( \gamma_1, \ldots, \gamma_k \) be the components of \( \gamma \setminus X \) lying in \( U_\infty \). Let \( \alpha_i = \Phi^{-1}(\gamma_i) \) be the pre-images of \( \gamma_i \) under \( \Phi \). Then every \( \alpha_i \) is a simple open arc in \( \{ z : |z| > 1 \} \) whose end points \( a_i, b_i \) are located on the unit circle; and all those open arcs \( \alpha_1, \ldots, \alpha_k \) are pairwise disjoint.

If \( k \geq 2 \), rename the arcs \( \alpha_2, \ldots, \alpha_k \) so that we can find an open arc \( \beta \subset (\mathbb{C} \setminus \overline{\mathbb{D}}) \) disjoint from \( \bigcup_{i=1}^k \alpha_i \) that connects a point \( a \) on \( \alpha_1 \) to a point \( b \) on \( \alpha_2 \). Then \( \gamma \cup \Phi(\beta) \) is a \( \Theta \)-curve separating \( x \) from \( y \) (see \([16, \text{Part B, Section VI}]\) for a definition of \( \Theta \)-curve). Let \( J_1 \) and \( J_2 \) denote the two components of \( \gamma \setminus \Phi(\beta) = \gamma \setminus \{ \Phi(a), \Phi(b) \} \). Then \( J_1 \cup \Phi(\beta) \) and \( J_2 \cup \Phi(\beta) \) are
both good cuts of $X$. One of them, denoted by $\gamma'$, separates $x$ from $y$ [16, \Theta-curve theorem, p.123]. By construction, this new good cut intersects $U_\infty$ at $k'$ open arcs for some $1 \leq k' \leq k - 1$. For relative locations of $J_1$, $J_2$ and $\Phi(\beta)$ in $\hat{C}$, we refer to Figure 2 in which $\gamma$ is represented as a circular circle, although a general good cut is usually not a circular circle. If $k' \geq 2$, we may use the same argument on $\gamma'$ and obtain a good cut $\gamma''$, that separates $x$ from $y$ and that intersects $U_\infty$ at $k''$ open arcs for some $1 \leq k'' \leq k - 2$. Repeating this procedure for at most $k - 1$ times, we will obtain a good cut separating $x$ from $y$ that intersects $U_\infty$ at a single open arc. \hfill \Box

**Theorem 6.4.** Let $X \subset \mathbb{C}$ be an unshielded continuum. Then every impression $\text{Imp}(\theta)$ is contained in a modified fiber $F_w^*$ for some $w \in \text{Imp}(\theta)$.

**Proof.** Suppose that a point $y \neq x$ on $\text{Imp}(\theta)$ is separated from $x$ in $X$ by a finite set. By Proposition 4.1, we can find a good cut $\gamma$ that separates $x$ from $y$. By Lemma 6.3, we may assume that $\gamma \cap U_\infty$ is an open arc $\gamma_1$. Let $a$ and $b$ be the two end points of $\alpha_1 = \Phi^{-1}(\gamma_1)$, an open arc in $\mathbb{C} \setminus \mathbb{D}$.

Fix an open arc $\mathcal{R}_\theta$ in $\mathbb{C} \setminus \mathbb{D}$ landing at the point $\theta$ such that $\Phi(\mathcal{R}_\theta) \cap X = \text{Imp}(\theta)$. We note that $\theta \in \{a, b\}$. Otherwise, there is a number $r > 1$ such that $\mathcal{R}_\theta \cap \{z : |z| < r\}$ lies in the component of

$$(\mathbb{C} \setminus \mathbb{D}) \setminus (\{a, b\} \cup \alpha_1)$$

whose closure contains $\theta$. From this we see that $\Phi(\mathcal{R}_\theta \cap \{z : |z| < r\})$ is disjoint from $\gamma$ and is entirely contained in one of the two components of $\mathbb{C} \setminus \gamma$, which contain $x$ and $y$ respectively. Therefore,

$$\Phi(\mathcal{R}_\theta \cap \{z : |z| < r\})$$

hence its subset $\text{Imp}(\theta)$ cannot contain $x$ and $y$ at the same time. This contradicts the assumption that $x, y \in \text{Imp}(\theta)$. 

![Figure 2](image-url)
Now we will lose no generality by assuming that \( e^{i\theta} = a \). Then \( \Phi(\mathcal{R}_\theta) \) intersects \( \gamma_1 \) infinitely many times, since \( \Phi(\mathcal{R}_\theta) \setminus \Phi(\mathcal{R}_\theta) \) contains \( \{x, y\} \). This implies that \( a \) is the landing point of \( \mathcal{R}_\theta \subset (\mathbb{C} \setminus \mathbb{D}) \).

Let \( w = \lim_{z \to a} \Phi|_{\alpha_1}(z) \). Then \( \{x, y, w\} \subset \text{Imp}(\theta) \), and the proof will be completed as soon as we verify that \( \text{Imp}(\theta) \subset F_w^* \).

Suppose there is a point \( w_1 \in \text{Imp}(\theta) \) that is not in \( F_w^* \). By Lemma 6.3 we may find a good cut \( \gamma'_1 \) separating \( w \) from \( w_1 \) that intersects \( U_\infty \) at an open arc \( \gamma'_1 \). Let \( \alpha'_1 = \Phi^{-1}(\gamma'_1) \). Since \( w \notin \gamma' \), the closure \( \overline{\alpha'_1} \) does not contain the point \( a \). Let \( I \) be the component of \( S^1 \setminus \overline{\alpha'_1} \) that contains \( a \). Then \( \mathcal{R}_\theta \cap \{z : |z| < r_1\} \) is disjoint from \( \alpha'_1 \) for some \( r_1 > 1 \). For such an \( r_1 \), the image \( \Phi(\mathcal{R}_\theta \cap \{z : |z| < r_1\}) \) is disjoint from \( \gamma'_1 \). On the other hand, the good cut \( \gamma' \) separates \( w \) from \( w_1 \). Therefore, the closure of \( \Phi(\mathcal{R}_\theta \cap \{z : |z| < r_1\}) \) hence its subset \( \text{Imp}(\theta) \) does not contain the two points \( w \) and \( w_1 \) at the same time. This is a contradiction. \( \square \)

**Remark 6.5.** Let \( J_c \) be the connected Julia set of a polynomial. The classes \( [x]_\sim \) obtained in this paper determine a monotone decomposition of \( J_c \), such that the quotient space is a locally connected continuum. Theorem 6.4 says that the impression of any prime end is entirely contained in a single class \( [x]_\sim \). Therefore, the finest decomposition mentioned in [1, Theorem 1] is finer than \( \{[x]_\sim : x \in J_c\} \). Currently it is not clear whether these two decompositions just coincide.

### 7. Facts and Examples

In this section, we give several examples to demonstrate the difference between (1) separating and cutting sets, (2) the modified fiber \( F_x^* \) and the class \( [x]_\sim \), (3) a continuum \( X \subset \mathbb{C} \) and the quotient space \( X/\sim \) for specific choices of \( X \). We also construct an infinite sequence of continua which have modified NLC-scales of any \( k \geq 2 \) and even up to \( \infty \), although the quotient of each of those continua is always homeomorphic with the unit interval \([0, 1]\).

**Example 7.1 (Separating Sets and Cutting sets).** For a set \( M \subset \mathbb{C} \), a set \( C \subset M \) is said to separate or to be a separating set between two points \( a, b \subset M \) if there is a separation \( M \setminus C = P \cup Q \) satisfying \( a \in P, b \in Q \); and a subset \( C \subset M \) is called a cutting set between two points \( a, b \in M \) if \( \{a, b\} \subset (X \setminus C) \) and if the component of \( X \setminus C \) containing \( a \) does not contain \( b \) [8, p.188,§47.VIII].

Let \( L_1 \) be the segment between the points \((2, 1)\) and \((2, 0)\) on the plane, \( Q_1 \) the one between \((-2, 0)\) and \( c = (0, \frac{1}{2}) \), and \( P_1 \) the broken line connecting...
(2, 0) to (−2, 0) through (0, −1), as shown in Figure 3. Define \((x_1, x_2) \xrightarrow{f} (\frac{1}{2}x_1, x_2)\). For any \(k \geq 1\), let \(L_{k+1} = g(L_k)\) and \(Q_{k+1} = g(Q_k)\); let \(P_{k+1} = f(P_k)\). Let \(B_k = L_k \cup P_k \cup Q_k\). Then \(\{B_k : k \geq 1\}\) is a sequence of broken lines converging to the segment \(B\) between \(a = (0, 0)\) and \(b = (0, 1)\). Let \(N = (\bigcup_k B_k) \bigcup B\). Then \(N\) is a continuum, which is not locally connected at each point of \(B\). Moreover, the singleton \(\{c\}\) is a cutting set, but not a separating set, between the points \(a\) and \(b\). The only nontrivial modified fiber is \(B = \{0\} \times [0, 1] = F^*_x\) for each \(x \in B\). So we have \(\ell^*(N) = 1\). Also, it follows that \([x]_\sim = B\) for all \(x \in B\) and \([x]_\sim = \{x\}\) otherwise. In particular, the broken lines \(B_k\) are still arcs in the quotient space but, under the metric of quotient space, their diameters converge to zero. Consequently, the quotient \(N/\sim\) is topologically the difference of a Hawaiian earring with a full open rectangle. See the right part of Figure 3. In other words, the quotient space \(N/\sim\) is homeomorphic with the quotient \(X/\sim\) of Example 7.2.

**Example 7.2 (The Witch’s Broom).** Let \(X\) be the witch’s broom [11, p.84, Figure 5.22]. See Figure 4. More precisely, let \(A_0 := [\frac{1}{2}, 1] \times \{0\}\); let
Let $A_k$ be the segment connecting $(1,0)$ to $(\frac{1}{2}, 2^{-k})$ for $k \geq 0$. Then $A = \bigcup_{k \geq 0} A_k$ is a continuum (an infinite broom) which is locally connected everywhere but at the points on $[\frac{1}{2}, 1) \times \{0\}$. Let $g(x) = \frac{1}{2}x$ be a similarity contraction on $\mathbb{R}^2$. Let $X = \{(0,0)\} \cup A \cup f(A) \cup f^2(A) \cup \cdots \cup f^n(A) \cup \cdots$. The continuum $X$ is called the *Witch’s Broom*. Consider the modified fibers of $X$, we have $F^*_x = \{x\}$ for each $x$ in $X \cap \{(x_1, x_2) : x_2 > 0\}$ and for $x = (0,0)$. The nontrivial modified fibers include: $F^*_{(1,0)} = [\frac{1}{2}, 1] \times \{0\}$, $F^*_{(2^{-k},0)} = [2^{-k-1}, 2^{-k+1}] \times \{0\}$ $(k \geq 1)$, and $F^*_{(x_1,0)} = [2^{-k}, 2^{-k+1}] \times \{0\}$ $(2^{-k} < x_1 < 2^{-k+1}, k \geq 1)$. Consequently, $[x]_\sim = \{x\}$ for each $x$ in $X \cap \{(x_1, x_2) : x_2 > 0\}$, while $[x]_\sim = [0,1] \times \{0\}$ for $x \in [0,1] \times \{0\}$. See the right part of Figure 4 for a depiction of the quotient $X/\sim$.

**Example 7.3 (Witch’s Double Broom).** Let $X$ be the witch’s broom. We call the union $Y$ of $X$ with a translated copy $X + (-1,0)$ the *witch’s double broom* (see Figure 5). Define $x \approx y$ if there exist points $x_1 = x, x_2, \ldots, x_n = y$ in $Y$ such that $x_i \in F^*_{x_{i-1}}$. Then $\approx$ is an equivalence and is not closed. Its closure $\approx^*$ is not transitive, since we have $(-1,0) \approx^* (0,0)$ and $(0,0) \approx^* (1,0)$, but $(-1,0)$ is not related to $(1,0)$ under $\approx^*$.

**Example 7.4 (Cantor’s Teepee).** Let $X$ be Cantor’s Teepee [14, p.145]. See Figure 6. Then the modified fiber $F^*_x = X$; and for every other point $x$, $F^*_x$ is exactly the line segment on $X$ that crosses $x$ and $p$. Therefore, $\ell^*(X) = 1$. Moreover, $[x]_\sim = X$ for every $x$, hence the quotient is a single point. In this case, we also say that $X$ is collapsed to a point.

**Example 7.5 (Cantor’s Comb).** Let $\mathcal{K} \subset [0,1]$ be Cantor’s ternary set. Let $X$ be the union of $\mathcal{K} \times [0,1]$ with $[0,1] \times \{1\}$. See Figure 7. We call
X the Cantor comb. Then the modified fiber $F^*_x = \{x\}$ for every point on $X$ that is off $\mathcal{K} \times [0, 1]$; and for every point $x$ on $\mathcal{K} \times [0, 1]$, the modified fiber $F^*_x$ is exactly the vertical line segment on $\mathcal{K} \times [0, 1]$ that contains $x$. Therefore, $\ell^*(X) = 1$. Moreover, $[x]_\sim = F^*_x$ for every $x$, hence the quotient is homeomorphic to $[0, 1]$. Here, we note that $X$ is locally connected at every point lying on the common part of $[0, 1] \times \{1\}$ and $\mathcal{K} \times [0, 1]$, although the modified fibers at those points are each a non-degenerate segment.

Example 7.6 (More Combs). We use Cantor’s ternary set $\mathcal{K} \subset [0, 1]$ to construct a sequence of continua $\{X_k : k \geq 1\}$, such that the modified NLC-scale $\ell^*(X_k) = k$ for all $k \geq 1$. We also determine the modified fibers and compute the quotient spaces $X_k/\sim$. Let $X_1$ be the union of $X'_1 = (\mathcal{K} + 1) \times [0, 2]$ with $[1, 2] \times \{2\}$. Here $\mathcal{K} + 1 := \{x_1 + 1 : x_1 \in \mathcal{K}\}$. Then $X_1$ is homeomorphic with Cantor’s Comb defined in Example 7.5. We have $\ell^*(X_1) = 1$ and that $X_1/\sim$ is homeomorphic with $[0, 1]$. Let $X_2$ be the union of $X_1$ with $[0, 1] \times (\mathcal{K} + 1)$. See Figure 8. Then the modified fiber of $X_2$ at the point $(1, 2) \in X_2$ is $F^*_{(1, 2)} = X_2 \cap \{(x_1, x_2) : x_1 \leq 1\}$, which will be referred to as the “largest modified fiber”, since it is the modified fiber in $X_2$ that has the largest modified NLC-scale. See the central part of Figure 8. The other modified fibers are either a single point or a segment, of the form $\{(x_1, x_2) : 0 \leq x_2 \leq 2\}$ for some $x_1 \in \mathcal{K} + 1$. Therefore, we have $\ell^*(X_2) = 2$ and can check that the quotient $X_2/\sim$ is homeomorphic with $[0, 1]$. Let $X_3$ be the union of $X_2$ with

$$\frac{X_1}{2} = \left\{\left(\frac{x_1}{2}, \frac{x_2}{2}\right) : (x_1, x_2) \in X_1\right\}.$$
Figure 8. A simple depiction of $X_2$, its largest modified fiber, and the quotient $X_2/\sim$.

Then the largest modified fiber of $X_3$ is exactly $F^{*}_{(1,2)} = X_3 \cap \{(x_1, x_2) : x_1 \leq 1\}$, which is homeomorphic with $X_2$. Therefore, $\ell^*(X_3) = 3$; moreover, $X_3/\sim$ is also homeomorphic with $[0,1]$. See upper part of Figure 9. Let $X_4 = X_2 \cup \frac{X_2}{2}$. Then the largest modified fiber of $X_4$ is $F^{*}_{(1,2)} = X_4 \cap \{(x_1, x_2) : x_1 \leq 1\}$, which is homeomorphic with $X_3$. Similarly, we can infer that $\ell^*(X_4) = 4$ and that $X_4/\sim$ is homeomorphic with $[0,1]$. See lower part of Figure 9. The construction of $X_k$ for $k \geq 5$ can be done inductively. The general formula $X_{k+2} = X_2 \cup \frac{1}{2}X_k$ defines a path-connected continuum for all $k \geq 3$, for which the largest modified fiber is homeomorphic to $X_{k+1}$. Therefore, we have $\ell^*(X_k) = k$; moreover, the quotient space $X_k/\sim$ is
always homeomorphic to the interval \([0, 1]\). Finally, we can verify that

\[ X_\infty = \{(0, 0)\} \cup \bigcup_{k=2}^{\infty} X_k \]

is a path connected continuum and that its largest modified fiber is homeomorphic to \(X_\infty\) itself. Therefore, \(X_\infty\) has a modified NLC-scale \(\ell^*(X_\infty) = \infty\), and the quotient \(X_\infty/\sim\) is also homeomorphic to \([0, 1]\).

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