MODIFIED FIBERS AND LOCAL CONNECTEDNESS OF PLANAR CONTINUA

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Abstract. We describe non-locally connected planar continua via the concepts of modified fiber and numerical scale.

Given a compactum $X \subset \mathbb{C}$ and $x \in \partial X$, we show that the set of points $y \in \partial X$ that cannot be separated from x by any finite set $C \subset \partial X$ is a continuum. This continuum is called the *modified fiber* F_x^* of X at x. If $x \in X^o$, we set $F_x^* = \{x\}$. When $F_x^* = \{x\}$ we show that the component of X containing x is locally connected at x. We also give an example of a planar continuum X, which is locally connected at a point $x \in X$ while the modified fiber F_x^* is not trivial.

The modified scale $\ell^*(X)$ of non-local connectedness is then the least integer p (or ∞ if such an integer does not exist) such that for each $x \in X$ there exist $k \leq p+1$ subcontinua

$$X = N_0 \supset N_1 \supset N_2 \supset \cdots \supset N_k = \{x\}$$

such that N_i is a modified fiber of N_{i-1} for $1 \leq i \leq k$. If $X \subset \mathbb{C}$ is an unshielded continuum or a continuum whose complement has finitely many components, we obtain that local connectedness of X is equivalent to the statement $\ell^*(X) = 0$.

We discuss the relation of our concepts to the works of Schleicher (1999) and Kiwi (2004). We further define an equivalence relation $\sim$ based on the modified fibers and show that the quotient space $X/\sim$ is a locally connected continuum. For connected Julia sets of polynomials and more generally for unshielded continua, we obtain that every prime end impression is contained in a modified fiber. Finally, we apply our results to examples from the literature and construct for each $n \geq 1$ examples of path connected continua X_n with $\ell^*(X_n) = n$.

1. Introduction and main results

Motivated by the construction of Yoccoz puzzles used in the study on local connectedness of quadratic Julia sets and the Mandelbrot set, Schleicher [13] introduces a notion of fiber for full continua (continua $M \subset \mathbb{C}$ having a connected complement $\mathbb{C} \setminus M$), based on "separation lines" chosen from particular countable dense sets of external rays that land on points of M. Kiwi [7] uses finite "cutting sets" to define another version of fiber for Julia sets, even when they are not connected.

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Jolivet-Loridant-Luo [5] replace Schleicher's "separation lines" with "good cuts", *i.e.*, simple closed curves J such that $J \cap \partial M$ is finite and $J \setminus M \neq \emptyset$. In this way, Schleicher's approach is generalized to continua $M \subset \mathbb{C}$ whose complement $\mathbb{C} \setminus M$ has finitely many components. For such a continuum M, the *pseudo-fiber* E_x (of M) at a point $x \in M$ is the collection of the points $y \in M$ that cannot be separated from x by a good cut; the *fiber* F_x *at* x is the component of E_x containing x. Here, a point y is separated from a point x by a simple closed curve J provided that x and y belong to different components of $\mathbb{C} \setminus J$. And the point x may belong to the bounded or unbounded component of $\mathbb{C} \setminus J$.

Clearly, the fiber F_x at x always contains x. We say that a fiber or a pseudo-fiber is *trivial* if it coincides with the single point set $\{x\}$.

By [5, Proposition 3.6], every fiber of M is again a continuum with finitely many complementary components. Thus the hierarchy by "fibers of fibers" is well defined. This allows to define the *scale* $\ell(M)$ *of non-local connectedness* as the least integer k such that for each $x \in M$ there exist $p \leq k+1$ subcontinua $M = N_0 \supset N_1 \supset \cdots \supset N_p = \{x\}$ such that N_i is a fiber of N_{i-1} for $1 \leq i \leq p$. If such an integer k does not exist, we set $\ell(M) = \infty$.

In this paper, we rather follow Kiwi's approach [7] and define "modified fibers" for compacta on the plane. The key point is: Kiwi focuses on Julia sets and uses "finite cutting sets" that consist of pre-periodic points, while we consider arbitrary compacta M on the plane (which may have interior points) and use "finite separating sets". We refer to Example 7.1 for the difference between separating and cutting sets. Moreover, in Jolivet-Loridant-Luo [5], a good cut is not contained entirely in the underlying continuum M. In the current paper, we will remove this assumption and only require that a good cut is a simple closed curve intersecting ∂M at finitely many points. After this slight modification we can establish the equivalence between the above mentioned two approaches, using good cuts or using finite separating sets. See Remark 1.3 for further details.

The notions and results will be presented in a way that focuses on the general topological aspects, rather than in the framework of complex analysis and dynamics.

Definition 1.1. Let $X \subset \mathbb{C}$ be a (possibly disconnected) compact set. We will say that a point $x \in \partial X$ is *separated from* $y \in \partial X$ by a (possibly empty) subset $C \subset X$ if there is a *separation* $\partial X \setminus C = A \cup B$ with $x \in A$ and $y \in B$.

Here " $\partial X \setminus C = A \cup B$ is a separation" means that $\overline{A} \cap B = A \cap \overline{B} = \emptyset$ (or, equivalently, that A and B are relatively open in $\partial X \setminus C$).

- The *modified fiber* F_x^* *of* X at a point x in the interior X^o of X is $\{x\}$; and the *modified fiber* F_x^* *of* X at a point $x \in \partial X$ is the set of the points $y \in \partial X$ that cannot be separated from x by any finite set $C \subset X$. Clearly, every F_x^* is closed in X; and the modified fiber of X at any $x \in \partial X$ equals that of ∂X at x. Actually, it is connected (Theorem 1).
- We inductively define a *modified fiber of order* $k \geq 2$ as a modified fiber of a continuum $Y \subset X$, where Y is a modified fiber of order $k-1$.
- The *local modified scale of non-local connectedness* of X at a point $x \in X$, denoted $\ell^*(X, x)$, is the least integer p such that there exist $k \leq p+1$ subcontinua
$$X = N_0 \supset N_1 \supset N_2 \supset \cdots \supset N_k = \{x\}$$
such that N_i is a modified fiber of N_{i-1} for $1 \leq i \leq k$. If such an integer does not exist we set $\ell^*(X, x) = \infty$.
- The (global) *modified scale of non-local connectedness* of X is
$$\ell^*(X) = \sup\{\ell^*(X, x) : x \in X\}.$$
We also call $\ell^*(X, x)$ the *modified NLC-scale of* X *at* x, and $\ell^*(X)$ the *modified NLC-scale of* X.

We will firstly obtain the connectedness of F_x^* and relate trivial modified fibers to local connectedness. Here, local connectedness at a particular point does not imply trivial modified fiber. In particular, let $\mathcal{K} \subset [0, 1]$ be Cantor's ternary set, let X be the union of $\mathcal{K} \times [0, 1]$ with $[0, 1] \times \{1\}$. See Figure 7. Then X is locally connected at every $x = (t, 1)$ with $t \in \mathcal{K}$, while the modified fiber F_x^* at this point is the whole segment $\{t\} \times [0, 1]$. See Example 7.5 for more details.

Theorem 1. *Let $X \subset \mathbb{C}$ be a compact set. Then F_x^* is connected for every $x \in X$. Moreover, $F_x^* = \{x\}$ implies that the component of X containing x is locally connected at x.*

Remark 1.2. This theorem is related to Kiwi's results [7, Corollaries 2.15 to 2.16] and can be considered as a generalization of these results to every compact set of the plane.

Secondly, we characterize modified fibers F_x^* through simple closed curves γ that separate x from points y in $X \setminus F_x^*$ and that intersect ∂X at a finite set or an empty set. This provides an equivalent way to develop the theory of modified fibers, for planar continua, and leads to a partial converse for the second part of Theorem 1. See Remark 1.3.

Theorem 2. *Let $x \in X$, where $X \subset \mathbb{C}$ is a continuum. Then, F_x^* consists of all points of X which cannot be separated from x by a simple closed curve intersecting ∂X in a finite set.*

This criterion is related to Kiwi's characterization of fibers [7, Corollary 2.18], as will be explained at the end of Section 4.

Remark 1.3. We define a simple closed curve γ to be a *good cut of a continuum* $X \subset \mathbb{C}$ if $\gamma \cap \partial X$ is a finite set (the empty set is also allowed). We also say that two points $x, y \in X$ are *separated by the good cut* γ if they lie in different components of $\mathbb{C}\backslash\gamma$. This slightly weakens the requirements on "good cuts" in [5]. Therefore, given a continuum $X \subset \mathbb{C}$ whose complement has finitely many components, the modified fiber F_x^* at any point $x \in X$ is a subset of the fiber F_x at x, if F_x is defined as in [5]. Consequently, we can infer that local connectedness of X implies triviality of all the modified fibers F_x^*, by citing two of the four equivalent statements of [5, Theorem 2.2]: (1) X is locally connected; (2) every fiber F_x is trivial. Combining this with Theorem 1, we see that every continuum $X \subset \mathbb{C}$ with $\mathbb{C} \setminus X$ having finitely many components is locally connected if and only if every modified fiber F_x^* of X is trivial. The same result does not hold when the complement $\mathbb{C} \setminus X$ has infinitely many components. Sierpinski's universal curve gives a counterexample. However, for the connected Julia set J of a polynomial (whose complement may have infinitely many components), we may set X to be the *filled Julia set*, *i.e.* the union of J and all the bounded components of $\mathbb{C} \setminus J$. Then the following holds.

(a) X is locally connected if and only if J is locally connected.
(b) The modified fiber of X at each $z \in X^o$ is just the singleton $\{z\}$ and the modified fiber of J at each $x \in J = \partial X$ is equal to the modified fiber of X at x.

The above item (a) is a direct corollary of [12, p.20, Theorem 2.1]. Therefore, the following statements also hold.

(c) Every modified fiber of X is trivial if and only if every modified fiber of J is trivial.
(d) J is locally connected if and only if $F_x^* = \{x\}$ for each $x \in J$.

Remark 1.4. The two approaches, via fibers F_x and via modified fibers F_x^*, have their own merits. The former one follows Schleicher's approach and is more closely related to the theory of puzzles in the study of Julia sets and the Mandelbrot set; hence it may be used to analyse the structure of such continua by cultivating the dynamics of polynomials. The latter approach has a potential to be extended to the study of general compact metric spaces; and, at the same time, it is directly connected with the first approach when restricted to planar continua.

Thirdly, we define an equivalence relation $\sim$ on X in Definition 1.5 and study the topology of X by investigating the quotient space $X/\!\sim$. When X is connected, this quotient is a locally connected continuum. See Theorem 3 below. For this relation $\sim$, every modified fiber F_x^* is contained in a single equivalence class.

Definition 1.5. Let $X \subset \mathbb{C}$ be a continuum. Let X_0 be the union of all the nontrivial modified fibers F_x^* for $x \in X$ and $\overline{X_0}$ denote the closure of X_0. We define $x \sim y$ if $x = y$ or if $x \neq y$ belong to the same component of $\overline{X_0}$.

It is easy to see that $\sim$ is an equivalence on X such that, for all $x \in X$, the equivalence class $[x]_\sim$ is a continuum containing the modified fiber F_x^*. Such a class equals $\{x\}$ if only $x \in (X \setminus \overline{X_0})$ or $\{x\}$ is a component of $\overline{X_0}$. Since the components of $\overline{X_0}$ form an upper semi-continuous decomposition of $\overline{X_0}$, one may further verify that $\{[x]_\sim : x \in X\}$ is also an upper semi-continuous decomposition. Hence $\sim$ is a closed equivalence and that the natural projection $\pi(x) = [x]_\sim$ is a monotone mapping, from X onto its quotient $X/\!\sim$. For the details we refer to Lemma 5.1.

Remark 1.6. Actually, there is a more natural equivalence relation $\approx$ by defining $x \approx y$ whenever there exist points $x_1 = x, x_2, \ldots, x_n = y$ in X such that $x_i \in F_{x_{i-1}}^*$. However, the relation $\approx$ may not be closed, as a subset of the product $X \times X$. On the other hand, if we take the closure of $\approx$ we will obtain a closed relation, which is reflexive and symmetric but may not be transitive (see Example 7.3). The above Definition 1.5 solves this problem.

The following theorem provides important information about the topology of $X/\!\sim$.

Theorem 3. *Let $X \subset \mathbb{C}$ be a continuum. Then $X/\!\sim$ is metrizable and is a locally connected continuum, possibly a single point.*

Remark 1.7. Kiwi [7] considers the special case where X is a component of the Julia set $J(f)$ of a polynomial f with degree ≥ 2 without irrationally neutral cycle. By Theorem 2, Corollary 1.1 and Theorem 3 of [7], the following interesting results concerning the structure of F_x^* hold.

(a) If $x \in J(f)$ is periodic or pre-periodic under f then $F_x^* = \{x\}$.
(b) If the modified fiber F_x^* of X at x is trivial then X is locally connected at x.
(c) The modified fiber F_x^* contains the impression(s) of at least one and at most finitely many prime ends and the impression of any prime end is contained in F_x^* if only it intersects F_x^*.

On the other hand, if X is the connected Julia set J of an arbitrary polynomial f the locally connected model introduced in [1] ensures that there is a finest monotone decomposition $\mathcal{D}$ of J such that the quotient space is a Peano continuum. In particular, every prime end impression is contained in a single element of $\mathcal{D}$. If f has no irrationally neutral cycle, from the results of [7] and [1] we can infer that every F_x^* lies in the single element of $\mathcal{D}$ that contains x. Those results are of fundamental significance from the viewpoint of topology. They also play a crucial role in the study of complex dynamics. Actually, the restriction $f|_J : J \to J$ induces a continuous map $\tilde{f} : \mathcal{D} \to \mathcal{D}$ such that $\pi \circ f = \tilde{f} \circ \pi$. Here $\pi : J \to \mathcal{D}$ is the natural projection sending $x \in J$ to the unique element of $\mathcal{D}$ that contains x. By Theorem 3, we also find some relations between modified fibers F_x^* and impressions of prime ends, when X is the Julia set of a polynomial (see Theorem 6.4). Combining this with laminations on the unit circle $S^1 \subset \mathbb{C}$, the system $f_\sim : J/\sim \to J/\sim$ is also a factor of the map $z \mapsto z^n$ on S^1. However, it is not known yet whether the decomposition $\{[x]_\sim : x \in X\}$ in classes of $\sim$ coincides with the finest decomposition $\mathcal{D}$ that corresponds to the locally connected model discussed in [1]. For more detailed discussions related to the dynamics of polynomials, we refer to [1, 7] and references therein.

Finally, to conclude the introduction, we propose the following problem.

Problem 1.8. To estimate the modified NLC-scale $\ell^*(X)$ from above for particular continua $X \subset \mathbb{C}$ such that $\mathbb{C} \setminus X$ has finitely many components, and to compare the quotient space $X/\sim$ with the locally connected model introduced in [1]. The Mandelbrot set or the Julia set of an infinitely renormalizable quadratic polynomial (when this Julia set is not locally connected) provide very typical choices for X. In particular, the modified NLC-scale $\ell^*(X)$ will be zero if the Mandelbrot set is locally connected, *i.e.*, if *MLC* holds. In such a case, the relation $\sim$ is trivial and its quotient is immediate.

Remark 1.9. Section 7 gives several examples of continua $X \subset \mathbb{C}$. For those continua X, we obtain the decomposition $\{[x]_\sim : x \in X\}$ into subcontinua and represent the quotient space $X/\sim$ on the plane. For those examples, the modified NLC-scale $\ell^*(X)$ is easy to determine.

Remark 1.10. The equivalence classes $[x]_\sim$ mentioned in Theorem 3 form a concrete upper semi-continuous decomposition of an arbitrary continuum X on the plane, with the property that the quotient space $X/\sim$ is a locally connected continuum. In the special case X is *unshielded*, *i.e*, X is equal to the boundary of the unbounded component of $\mathbb{C} \setminus X$, the finest decomposition in [1, Theorem 1] is finer than or equal to our decomposition $\{[x]_\sim : x \in X\}$. We refer to Theorem 6.4 for details when X is assumed to be unshielded. It is not known whether those two decompositions actually coincide. If the answer is yes, the quotient space $X/\sim$ in Theorem 3 is exactly the finest locally connected model of X, which shall be in some sense "computable". Here, an application of some interest is to study the locally connected model of an infinitely renormalizable Julia set [4] or of the Mandelbrot set, as mentioned in Problem 1.8.

We arrange our paper as follows. Section 2 recalls some basic notions and results from topology that are closely related to local connectedness. Sections 3, 4 and 5 respectively prove Theorems 1, 2 and 3. Section 6 discusses basic properties of modified fibers, studies those continua from a viewpoint of dynamic topology (as proposed by Whyburn and Duda [16, pp.130-144]) and relates the theory of modified fibers to the theory of prime ends for unshielded continua. Finally, in Section 7, we illustrate our results through examples from the literature and give an explicit sequence of path connected continua X_n satisfying $\ell^*(X_n) = n$.

2. A Revisit to Local Connectedness

Definition 2.1. A topological space X is *locally connected at* a point $x_0 \in X$ if for any neighborhood U of x_0 there exists a connected neighborhood V of x_0 such that $V \subset U$, or equivalently, if the component of U containing x_0 is also a neighborhood of x_0. The space X is then called *locally connected* if it is locally connected at every of its points.

We focus on metric spaces and their subspaces. The following characterization can be found as the definition of locally connectedness in [16, Part A, Section XIV].

Lemma 2.2. *A metric space (X,d) is locally connected at $x_0 \in X$ if and only if for any $\varepsilon > 0$ there exists $\delta > 0$ such that any point $y \in X$ with $d(x_0,y) < \delta$ is contained together with x_0 in a connected subset of X which is of diameter less than ε.*

When X is compact, Lemma 2.2 is a local version of [9, p.183, Lemma 17.13(d)]. For the convenience of the readers, we give here the concrete statement as a lemma.

Lemma 2.3. *A compact metric space X is locally connected if and only if for every $\varepsilon > 0$ there is $\delta > 0$ so that any two points of distance less than δ are contained in a connected subset of X which is of diameter less than ε.*

Using Lemma 2.2, we obtain a fact concerning continua of the Euclidean space $\mathbb{R}^n$.

Lemma 2.4. *Let $X \subset \mathbb{R}^n$ be a continuum and $U = \bigcup_{\alpha\in I} W_\alpha$ the union of any collection $\{W_\alpha : \alpha \in I\}$ of components of $\mathbb{R}^n \setminus X$. If X is locally connected at $x_0 \in X$, then so is $X \cup U$. Consequently, if X is locally connected, then so is $X \cup U$.*

Proof. Choose δ with properties from Lemma 2.2 with respect to x_0, X and $\varepsilon/2$. For any $y \in U$ with $d(x_0,y) < \delta$ we consider the segment $[x_0,y]$ between x_0 and y. If $[x_0,y] \subset (X \cup U)$, we are done. If not, choose the point $z \in ([x_0,y] \cap X)$ that is closest to y. Since $y \in U$ lies in a component W_α of $\mathbb{R}^n \setminus X$ and since $W_\alpha \subset U$, the segment $[y,z]$ is contained in $\overline{W_\alpha}$ and hence is a subset of $X \cup U$. By the choice of δ and Lemma 2.2, we may connect z and x_0 with a continuum $A \subset X$ of diameter less than $\varepsilon/2$. Therefore, the continuum $B := A \cup [y,z] \subset (X \cup U)$ is of diameter at most ε as desired. $\square$

In the present paper, we are mostly interested in continua on the plane, especially continua X which are on the boundary of a continuum $M \subset \mathbb{C}$. Typical choice of such a continuum M is the connected filled Julia set of a rational function. Several results from [16] will be very helpful in our study.

The first result gives a fundamental fact about a continuum failing to be locally connected at one of its points. The proof can be found in [16, p.124, Corollary].

Lemma 2.5. *If a continuum M is not locally connected at a point p then it is not locally connected at all points of a nondegenerate subcontinuum of M that contains p.*

The second result will be referred to as **Torhorst Theorem** (see [16, p.124, Torhorst Theorem] and [16, p.126, Lemma 2]).

Lemma 2.6. *The boundary B of each component C of the complement of a locally connected continuum M is itself a locally connected continuum. If further M has no cut point, then B is a simple closed curve.*

We finally recall a *Plane Separation Theorem* [16, p.120, Exercise 2].

Proposition 2.7. *If A is a continuum and B is a closed connected set of the plane with $A \cap B$ being a totally disconnected set, and with $A \setminus B$ and $B \setminus A$ being connected, then there exists a simple closed curve J separating $A \setminus B$ and $B \setminus A$ such that $J \cap (A \cup B) \subset (A \cap B)$.*

3. Fundamental properties of modified fibers

The proof for Theorem 1 has two parts. We start from the connectedness of modified fibers F_x^*. Since $F_x^* = \{x\}$ for all $x \in X^o$, we only consider the modified fibers F_x^* for $x \in \partial X$, which just equals the modified fiber of ∂X at x.

Theorem 3.1. *Let $X \subset \mathbb{C}$ be a compact set. Then every F_x^* is connected.*

Proof. Suppose on the contrary that F_x^* is disconnected for some $x \in \partial X$. Then we can fix a separation $F_x^* = A \cup B$ with $x \in A$ and $B \neq \emptyset$. Fix a point $x' \in B$. Because A and B are compact and disjoint nonempty sets, they have some open neighbourhoods A^* (of A) and B^* (of B) with disjoint closures. Then $K = \partial X \setminus (A^* \cup B^*)$ is a compact subset of ∂X. As $F_x^* \cap K = \emptyset$, we may find for each $z \in K$ a finite set C_z and a separation

$$\partial X \setminus C_z = U_z \cup V_z$$

such that $x \in U_z, z \in V_z$ and both of U_z and V_z are relatively open in ∂X. By flexibility of $z \in K$, we obtain an open cover $\{V_z : z \in K\}$ of K, which then has a finite subcover $\{V_{z_1}, \ldots, V_{z_n}\}$. Let

$$U = U_{z_1} \cap \cdots \cap U_{z_n}, \ V = V_{z_1} \cup \cdots \cup V_{z_n}.$$

Then U, V are disjoint sets open in ∂X such that $C := \partial X \setminus (U \cup V)$ is a subset of $C_{z_1} \cup \cdots \cup C_{z_n}$, hence it is also a finite set.
Let $C' = C \setminus \{x'\}$. Then:

$$\partial X \setminus C' = U \cup \{x'\} \cup V = (U \cap A^*) \cup ((U \cap B^*) \cup \{x'\} \cup V)$$

and the right-hand side is a separation with $x \in U \cap A^*$ and $x' \in (U \cap B^*) \cup \{x'\} \cup V$. (This separation follows from the fact that each of the sets $(U \cap B^*) \cup \{x'\}$ and V is separated from $U \cap A^*$, since U is separated from V and A^* from B^*, and hence from $\{x'\}$). This contradicts the assumption that $x' \in F_x^*$, because F_x^* being the modified fiber at x, none of its points can be separated from x by the finite set C'. □

Then we recover in fuller generality that triviality of the modified fiber at a point x in a continuum $X \subset \mathbb{C}$ implies local connectedness of X at x. More restricted versions of this result appear earlier: in [13] for continua in the plane with connected complement, in [7] for Julia sets of monic polynomials or the components of such a set, and in [5] for continua in the plane whose complement has finitely many components. In the remaining part of this section, we denote by $B(x, r)$ the open disk centered at x with radius $r > 0$.

Theorem 3.2. *If $F_x^* = \{x\}$ for a point x in a continuum $X \subset \mathbb{C}$ then X is locally connected at x.*

Proof. We only consider the case $x \in \partial X$ and prove that if X is not locally connected at x then F_x^* contains a non-degenerate continuum $M' \subset \partial X$.

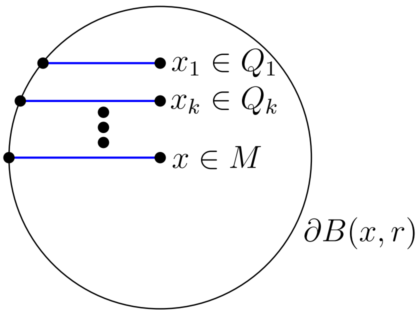

FIGURE 1. The components Q_k and the limit $M \subset Q_x$.

By definition, if X is not locally connected at x there exists a number $r > 0$ such that the component Q_x of $\overline{B(x,r)} \cap X$ containing x is not a neighborhood of x in X. This means that there exists a sequence of points $\{x_k\}_{k=1}^{\infty} \subset (B(x,r) \cap X) \setminus Q_x$ such that $\lim\limits_{k\to\infty} x_k = x$. Let Q_k be the component of $\overline{B(x,r)} \cap X$ containing x_k. By [11, p.74, Boundary Bumping Theorem II] applied to X and its proper subset $\overline{B(x,r)} \cap X$, it follows that Q_x as well as all the sets Q_k intersect $\partial B(x,r)$. Moreover, $Q_i \cap \{x_k\}_{k=1}^{\infty}$ is a finite set for each $i \geq 1$; hence we may assume, by taking a subsequence, that $Q_i \cap Q_j = \emptyset$ for $i \neq j$.

Since the nonempty compact subsets of X form a compact metric space under Hausdorff distance, we may further assume that there exists a continuum M such that $\lim\limits_{k\to\infty} Q_k = M$ under Hausdorff distance. Clearly, we have $x \in M \subset Q_x$. Note that Q_x, and thus M, may not be contained in ∂X. In the sequel, we will obtain a non-degenerate sub-continuum $M' \subset M$ with $x \in M' \subset \partial X$.

Let W_k be the unbounded component of $\mathbb{C} \setminus Q_k$. Then $x \in W_k$ for all $k \geq 1$ and every ∂W_k is a continuum intersecting $\partial B(x,r)$. Note that it is possible that $\partial B(x,r) \cap \partial W_k \cap X^o \neq \emptyset$. Hence, let us set $E_k := B(x,r) \cap \partial W_k$. Then $E_k \subset \partial X$. Indeed, by definition of W_k, we have $\partial W_k \subset \partial Q_k \subset Q_k$. Therefore, if $y \in E_k \cap X^o$, then $y \in B(x,r) \cap X^o \cap Q_k$, hence $y \in Q_k^o$, a contradiction to $y \in \partial Q_k$.

Then, we consider the segment $\overline{xx_k}$ and denote by y_k the last point (from $x_k \in Q_k$ to x) of this segment that also lies on Q_k. Note that $y_k \in E_k$ for all $k \geq 1$ and that $\lim\limits_{k\to\infty} y_k = \lim\limits_{k\to\infty} x_k = x$.

Applying [11, p.74, Boundary Bumping Theorem II] to ∂W_k and its proper subset E_k, we see that the closure of the component Q'_k of E_k containing y_k intersects $\partial B(x,r)$. Since $E_k \subset \partial X$, we have $\overline{Q'_k} \subset \partial X$ and $\overline{Q'_k} \cap \partial B(x,r) \neq \emptyset$. Therefore, we may fix an appropriate sub-sequence of $\left\{\overline{Q'_k}\right\}$ converging to a limit continuum M' under Hausdorff distance.

Clearly, every continuum $\overline{Q'_k}$ intersects $\partial B(x,r)$, so does the limit M'. On the other hand, as $y_k \in Q'_k \subset \partial X$ and $\lim\limits_{k\to\infty} y_k = x$, we also have $x \in M' \subset \partial X$. In particular, M' is non-degenerate. Moreover, no point $y \in M' \setminus \{x\}$ can be separated from x in ∂X by a finite set C, since $\partial X \setminus C$ includes $(M' \cup \bigcup_{k\geqslant 1} \overline{Q'_k}) \setminus C$ as a subset, which contains a sub-sequence of the above continua $\overline{Q'_k}$ converging to $M' \supset \{x,y\}$. This indicates that F_x^* contains the non-degenerate continuum M' thus is not trivial. ☐

Note that Theorem 3.1 proves the first statement of Theorem 1. For the second statement of Theorem 1, assuming that the modified fiber of X at x is trivial implies that the modified fiber of the component Q_x of X containing x is also trivial. We can then apply Theorem 3.2 to Q_x.

4. Schleicher's and Kiwi's approaches unified

The following proposition implies Theorem 2. We present it in this form, since it can be seen as a modification of the plane separation theorem (Proposition 2.7).

Proposition 4.1. *Let C be a finite subset of a continuum $X \subset \mathbb{C}$ and x, y two points on $X \setminus C$. If there is a separation $X \setminus C = P \cup Q$ with $x \in P$ and $y \in Q$ then x is separated from y by a simple closed curve γ with $(\gamma \cap X) \subset C$.*

Proof. By [11, p.73, Boundary Bumping Theorem I], we know that every component of $\overline{P}$ intersects C. Now, since C is a finite set, it follows that $\overline{P}$ has finitely many components, say $P_1, \ldots, P_k$. We may assume that $x \in P_1$.

Similarly, every component of $\overline{Q} \subset (Q \cup C)$ intersects C and $\overline{Q}$ has finitely many components, say $Q_1, \dots, Q_l$. We may assume that $y \in Q_1$.

Let $P_1^* = P_2 \cup \cdots \cup P_k \cup Q_1 \cup \cdots \cup Q_l$. Then $X = P_1 \cup P_1^*$, $x \in P_1$, $y \in P_1^*$ and $(P_1 \cap P_1^*) \subset C$. Let $N_1 = \{z \in P_1;\ \mathrm{dist}(z, P_1^*) \geq 1\}$ and for each $j \geq 2$, let [1]

$$N_j = \{z \in P_1 :\ 3^{-j} \leq \mathrm{dist}(z, P_1^*) \leq 3^{-j+1}\}.$$

Clearly, every N_j is a compact set. Therefore, we may cover N_j by finitely many open disks centered at a point in N_j and with radius $r_j = 3^{-j-1}$, say $B(x_{j1}, r_j), \dots, B(x_{jk(j)}, r_j)$.

For $j \geqslant 1$, let us set $M_j = \bigcup_{i=1}^{k(j)} \overline{B(x_{ji}, r_j)}$. Then $M = \overline{\bigcup_{j\geqslant 1} M_j}$ is a compact set containing P_1. Its interior M^o contains x. Moreover, $P_1^* \cap \left(\bigcup_{j\geqslant 1} M_j\right) = \emptyset$ by definition of N_j and M_j, while $M \setminus \left(\bigcup_j M_j\right)$ is a subset of $P_1 \cap P_1^*$, hence we have $M \cap P_1^* = P_1 \cap P_1^*$ and $y \notin M$. Also, $\partial M \cap X$ is a subset of $P_1 \cap P_1^*$, hence it is a finite set.

Now M is a continuum, since P_1 is itself a continuum and the disks $B(x_{ji}, r_j)$ are centered at $x_{ji} \in N_j$. The continuum M is even locally connected at every point on $M \setminus C = \bigcup_j M_j$. Indeed, it is locally a finite union of disks, since $M_j \cap M_k = \emptyset$ as soon as $|j - k| > 1$ and since every point of $M \setminus C$ is in one of these disks. As C is finite, it follows from Lemma 2.5 that M is a locally connected continuum.

Now, let U be the component of $\mathbb{C} \setminus M$ that contains y. By Torhorst Theorem, see Lemma 2.6, the boundary ∂U of U is a locally connected continuum. Therefore, by Lemma 2.4, the union $U \cup \partial U$ is also a locally connected continuum. Since U is a complementary component of ∂U, the union $U \cup \partial U$ even has no cut point. It follows from Torhorst Theorem that the boundary ∂V of any component V of $\mathbb{C} \setminus (U \cup \partial U)$ is a simple closed curve. Note that this curve separates every point of U from any point of V. Choosing V to be the component of $\mathbb{C} \setminus (U \cup \partial U)$ containing x, we obtain a simple closed curve $J = \partial V$ separating y from x.

Finally, since $J = \partial V \subset \partial U \subset \partial M$, we see that $J \cap X$ is contained in the finite set C. Consequently, J is a good cut of X separating x from y. $\square$

This result proves Theorem 2 and is related to Kiwi's characterization of fibers. Restricting to connected Julia sets $J(f)$ of polynomials f, Kiwi [7] defines for $\zeta \in J(f)$ the fiber Fiber(ζ) as the set of $\xi \in J(f)$ such that ξ and ζ lie in the same connected component of $J(f) \setminus Z$ for every finite set $Z \subset J(f)$, made up of periodic or preperiodic points that are not in the grand orbit of a Cremer point. Here we want to note that $J(f) \setminus Z$ always

[1] This idea is inspired by the proof of the plane separation theorem, see Proposition 2.7

has finitely many components. Kiwi showed in [7, Corollary 2.18] that these fibers Fiber(ζ) can be characterized by using separating curves involving external rays.

5. A locally connected model for the continuum X

This section recalls a few notions and results from Kelley's *General Topology* [6] and proves Theorem 3 in two steps:

(1) $X/\sim$ is metrizable, hence is a compact connected metric space, *i.e.*, a continuum.
(2) $X/\sim$ is a locally connected continuum.

A decomposition $\mathcal{D}$ of a topological space X is *upper semi-continuous* if for each $D \in \mathcal{D}$ and each open set U containing D there is an open set V such that $D \subset V \subset U$ and V is the union of members of $\mathcal{D}$ [6, p.99]. Given a decomposition $\mathcal{D}$, we may define a projection $\pi : X \to \mathcal{D}$ by setting $\pi(x)$ to be the unique member of $\mathcal{D}$ that contains x. Then, the quotient space $\mathcal{D}$ is equipped with the largest topology such that $\pi : X \to \mathcal{D}$ is continuous.

By [11, p.40, Theorem 3.9], any upper semi-continuous decomposition of a compact metric space is metrizable. Therefore, the following lemma ensures that $X/\sim$ is compact and metrizable.

Lemma 5.1. *The decomposition* $\{[x]_\sim : x \in X\}$ *is upper semi-continuous.*

Proof. Let X_0 be the union of all the nontrivial F_x^*. In particular, $\overline{X_0} \subset \partial X$. Clearly, the above decomposition $\{[z]_\sim : x \in X\}$ consists of all components of the closed subset $\overline{X_0}$ of the compact space X, and of all singletons in $X \setminus \overline{X_0}$. Hence the assertion follows from the well known fact that such a decomposition is upper semi-continuous. □

Theorem 5.2. *The quotient* $X/\sim$ *is a locally connected continuum.*

Proof. As X is a continuum, $\pi(X) = X/\sim$ is itself a continuum. We now prove that this quotient is locally connected. If V is an open set in $X/\sim$ that contains $[x_0]_\sim$, as an element of $X/\sim$, then the pre-image $U := \pi^{-1}(V)$ is open in X and contains the class $[x_0]_\sim$ as a subset. We shall prove that V contains a connected neighborhood of $[x_0]_\sim$.

We assume $U \neq X$ and let Q be the component of U containing $[x_0]_\sim$. Then, applying [11, p.74, Boundary Bumping Theorem II] to $U \subset X$ we have $\overline{Q} \setminus U \neq \emptyset$. Note that we only need to verify that Q is also a neighborhood of $[x_0]_\sim$, since the monotonicity of the projection $\pi(z) = [z]_\sim$ will then ensure that Q contains a saturated neighborhood of x_0, say W, whose image under π is a neighborhood of the point $[x_0]_\sim$ in V.

Suppose on the contrary that Q is not a neighborhood of $[x_0]_\sim$. Then we can find an infinite sequence $\{x_k\}$ in $U \setminus Q$ with $\lim\limits_{k\to\infty} x_k = x \in [x_0]_\sim$. Here the component Q_k of U containing x_k satisfies $Q_k \cap Q = \emptyset$. And we may assume that $Q_i \cap Q_j = \emptyset$ for $i \neq j$. Applying[11, p.74, Boundary Bumping Theorem II] to the continuum X and its proper open subset U, we have $\overline{Q_k} \setminus U = \overline{Q_k} \cap \partial U \neq \emptyset$.

Cover the compact set $X \setminus U$ by finitely many open disks $D_1, \ldots, D_n$ with centers $y_i \in (X \setminus U)$ and radius $r < \frac{1}{3} r_0$, where r_0 is the distance between the closed sets $X \setminus U$ and $[x_0]_\sim$. Those disks D_i may be chosen so that $|y_i - y_j| \neq 2r$ for all $i \neq j$. Then $Y = \bigcup_i \overline{D_i}$ is disjoint from $[x_0]_\sim$ and every of its components is a continuum having no cut point.

By Torhorst Theorem, cited as Lemma 2.6 in Section 2 of this paper, the boundary ∂Y consists of finitely many Jordan curves. The choice of the disks $\{D_i\}$ implies that those Jordan curves are pairwise disjoint. Hence the component U_Y of $\mathbb{C} \setminus Y$ containing x is topologically equivalent to a circle domain, *i.e.*, the difference between the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and finitely many pairwise disjoint closed disks contained in $\mathbb{D}$.

By the choices of the disks $D_1, \ldots, D_n$ we can infer that $Y \cap [x_0]_\sim = \emptyset$. Thus we have $[x_0]_\sim \subset U_Y$. By truncating finitely many x_i we may assume that $x_k \in U_Y$ for all $k \geq 1$. Since the continua $\overline{Q_1}, \overline{Q_2}, \ldots,$ all intersect $\partial U \subset Y \subset (\mathbb{C} \setminus U_Y)$, there is a component Γ of ∂U_Y such that $\overline{Q_k} \cap \Gamma \neq \emptyset$ for infinitely many (and, *w.l.o.g.*, for all) $k \geq 1$. This Γ is a Jordan curve, because U_Y is a circle domain.

By Schönflies Theorem [10, pp.71-72, Theorem 3 and 4], we may fix a homeomorphism $\varphi : \mathbb{C} \to \mathbb{C}$ sending the component of $\mathbb{C} \setminus \Gamma$ that contains x (and hence $[x_0]_\sim$) onto the open unit disk $\mathbb{D}$. For all $k \geq 1$ let Q'_k be the component of $\overline{Q_k} \cap U_Y$ containing x_k. Now applying [11, p.74, Boundary Bumping Theorem II] to the continuum $\overline{Q_k}$ and its proper open subset $\overline{Q_k} \cap U_Y$, we know that $\overline{Q'_k} \cap \Gamma \neq \emptyset$. For the same reasons, the component Q' of $\overline{Q} \cap U_Y$ containing x satisfies $\overline{Q'} \cap \Gamma \neq \emptyset$.

Let W_k be the unbounded component of $\mathbb{C} \setminus \varphi\left(\overline{Q'_k}\right)$. Then ∂W_k is a continuum intersecting $\partial\mathbb{D} = \varphi(\Gamma)$ such that $E_k = \mathbb{D} \cap \partial W_k \subset \varphi(\partial X)$. Here it is possible that $\partial\mathbb{D} \cap \partial W_k \cap \varphi(X^o) \neq \emptyset$. Moreover, we have $\varphi(x) \in W_k$ for all $k \geq 1$, since $\varphi\left(\overline{Q'}\right)$ can be connected to infinity by an external ray of $\overline{\mathbb{D}}$ landing on a point in $\varphi\left(\overline{Q'}\right) \cap \partial\mathbb{D}$.

Let y_k be the last point lying on $\varphi\left(\overline{Q_k}\right)$ from $\varphi(x_k)$ to $\varphi(x)$ along the segment $\overline{\varphi(x_k)\varphi(x)} \subset \mathbb{D}$. Then $y_k \in E_k$ for all $k \geq 1$; moreover, we have $\lim\limits_{k\to\infty} \varphi^{-1}(y_k) = x$. Let Q''_k be the component of E_k containing y_k. Then

Boundary Bumping Theorem II [11, p.74] ensures that $\overline{Q''_k}$ is a continuum that intersects $\partial\mathbb{D}$. Thus we have $\varphi^{-1}\left(\overline{Q''_k}\right) \cap \Gamma \neq \emptyset$ for all $k \geq 1$.

Finally, by the containments $\varphi^{-1}\left(\overline{Q''_k}\right) \subset \varphi^{-1}\left(\overline{E_k}\right) \subset \partial X$, we may choose a subsequence $M_i = \varphi^{-1}\left(\overline{Q''_{k(i)}}\right)$ that converges to a limit continuum M' under Hausdorff distance. Then we have $x \in M' \subset [x_0]_\sim$ and $M' \cap \Gamma \neq \emptyset$. This is absurd, since we have $[x_0]_\sim \subset U_Y$ and $\Gamma \subset \partial U_Y$. $\square$

6. How modified fibers are changed under continuous maps

This section discusses how modified fibers are changed under continuous maps. As a special application, we may compare the dynamics of a polynomial $f_c(z) = z^n + c$ on its Julia set J_c, the expansion $z \mapsto z^d$ on unit circle, and an induced map $\tilde{f}_c$ on the quotient $J_c/\sim$.

Let $X, Y \subset \mathbb{C}$ be continua and $x \in X$ a point. The first primary observation is that $f(F_x^*) \subset F_{f(x)}^*$ for any finite-to-one continuous surjection $f: X \to Y$.

Indeed, for any $y \neq x$ in the modified fiber F_x^* and any finite set $C \subset Y$ that is disjoint from $\{f(x), f(y)\}$, we can see that $f^{-1}(C)$ is a finite set disjoint from $\{x, y\}$. Since $y \in F_x^*$ there exists no separation $X \setminus f^{-1}(C) = A \cup B$ with $x \in A, y \in B$; therefore, there exists no separation $Y \setminus C = P \cup Q$ with $f(x) \in P, f(y) \in Q$. This certifies that $f(y) \in F_{f(x)}^*$.

The inclusion $f(F_x^*) \subset F_{f(x)}^*$ indicates that $f(X_0) \subset Y_0$, where X_0 is the union of all the nontrivial modified fibers F_x^* in X, and Y_0 the union of those in Y. It follows that $f([x]_\sim) \subset [f(x)]_\sim$. Therefore, the correspondence $[x]_\sim \xrightarrow{\tilde{f}} [f(x)]_\sim$ gives a well defined map $\tilde{f}: X/\sim \to Y/\sim$ that satisfies the following commutative diagram, in which each downward arrow $\downarrow$ indicates the natural projection from a space onto its quotient.

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ \downarrow & & \downarrow \\ X/\sim & \xrightarrow{\quad \tilde{f} \quad} & Y/\sim \end{array}$$

Given an open set $U \subset Y/\sim$, we can use the definition of quotient topology to infer that $V := \tilde{f}^{-1}(U)$ is open in $X/\sim$ whenever $\pi^{-1}(V)$ is open in X. On the other hand, the above diagram ensures that $\pi^{-1}(V) = f^{-1}\left(\pi^{-1}(U)\right)$, which is an open set of X, by continuity of f and π.

The above arguments lead us to a useful result for the study of dynamics on Julia sets.

Theorem 6.1. *Let $X, Y \subset \mathbb{C}$ be continua. Let the relation $\sim$ be defined as in Theorem 3. If $f : X \to Y$ is continuous, surjective and finite-to-one then $\tilde{f}([x]_\sim) := [f(x)]_\sim$ defines a continuous map with $\pi \circ f = \tilde{f} \circ \pi$.*

Remark 6.2. Every polynomial $f_c(z)$ restricted to its Julia set J_c satisfies the conditions of Theorem 6.1, if we assume that J_c is connected; so the restricted system $f_c : J_c \to J_c$ has a factor system $\tilde{f}_c : J_c/\sim \to J_c/\sim$, whose underlying space is a locally connected continuum.

Let $X \subset \mathbb{C}$ be an unshielded continuum. Let U_∞ be the unbounded component of $\mathbb{C} \setminus X$. Here, X is unshielded provided that $X = \partial U_\infty$. Let $\mathbb{D} := \{z \in \hat{\mathbb{C}} : |z| < 1\}$ be the unit disk. By Riemann Mapping Theorem, there exists a conformal isomorphism $\Phi : \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to U_\infty$ that fixes ∞ and has positive derivative at ∞. The prime end theory [2, 15] builds a correspondence between an angle $\theta \in S^1 := \partial\mathbb{D}$ and a continuum

$$Imp(\theta) := \left\{ w \in X : \exists \, z_n \in \mathbb{D} \text{ with } z_n \to e^{\mathbf{i}\theta}, \lim_{n\to\infty} \Phi(z_n) = w \right\}$$

We call $Imp(\theta)$ the *impression of* θ. By [3, p.173, Theorem 9.4], we may fix a simple open arc $\mathcal{R}_\theta$ in $\mathbb{C}\setminus\overline{\mathbb{D}}$ landing at the point $e^{\mathbf{i}\theta}$ such that $\overline{\Phi(\mathcal{R}_\theta)} \cap X = Imp(\theta)$.

We will connect impressions to modified fibers. Before that, we obtain a useful lemma concerning good cuts of an unshielded continuum X on the plane. Here a good cut of X is a simple closed curve that intersects X at a finite subset (see Remark 1.3).

Lemma 6.3. *Let $X \subset \mathbb{C}$ be an unshielded continuum and U_∞ the unbounded component of $\mathbb{C} \setminus X$. Let x and y be two points on X separated by a good cut γ of X. Then we can find a good cut separating x from y that intersects U_∞ at an open arc.*

Proof. Since each of the two components of $\mathbb{C} \setminus \gamma$ intersects $\{x, y\}$, we have $\gamma \cap U_\infty \neq \emptyset$. Since $\gamma \cap X$ is a finite set, the difference $\gamma \setminus X$ has finitely many components. Let $\gamma_1, \ldots, \gamma_k$ be the components of $\gamma \setminus X$ lying in U_∞. Let $\alpha_i = \Phi^{-1}(\gamma_i)$ be the pre-images of γ_i under Φ. Then every α_i is a simple open arc in $\{z : |z| > 1\}$ whose end points a_i, b_i are located on the unit circle; and all those open arcs $\alpha_1, \ldots, \alpha_k$ are pairwise disjoint.

If $k \geqslant 2$, rename the arcs $\alpha_2, \ldots, \alpha_k$ so that we can find an open arc $\beta \subset (\mathbb{C} \setminus \overline{\mathbb{D}})$ disjoint from $\bigcup_{i=1}^k \alpha_i$ that connects a point a on α_1 to a point b on α_2. Then $\gamma \cup \Phi(\beta)$ is a Θ-curve separating x from y (see [16, Part B, Section VI] for a definition of Θ-curve). Let J_1 and J_2 denote the two components of $\gamma \setminus \overline{\Phi(\beta)} = \gamma \setminus \{\Phi(a), \Phi(b)\}$. Then $J_1 \cup \overline{\Phi(\beta)}$ and $J_2 \cup \overline{\Phi(\beta)}$ are both good cuts of X. One of them, denoted by γ', separates x from y [16, Θ-curve theorem, p.123]. By construction, this new good cut intersects U_∞ at k' open arcs for some $1 \leqslant k' \leqslant k-1$. For relative locations of J_1, J_2 and $\Phi(\beta)$ in $\hat{\mathbb{C}}$, we refer to Figure 2 in which γ is represented as a circular circle, although a general good cut is usually not a circular circle. If $k' \geqslant 2$, we

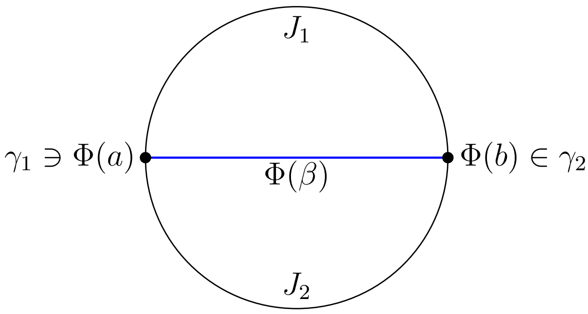

FIGURE 2. The Θ-curve together with the arcs J_1, J_2, and $\Phi(\beta)$.

may use the same argument on γ' and obtain a good cut γ'', that separates x from y and that intersects U_∞ at k'' open arcs for some $1 \leqslant k'' \leqslant k-2$. Repeating this procedure for at most $k-1$ times, we will obtain a good cut separating x from y that intersects U_∞ at a single open arc. $\square$

Theorem 6.4. *Let $X \subset \mathbb{C}$ be an unshielded continuum. Then every impression $Imp(\theta)$ is contained in a modified fiber F_w^* for some $w \in Imp(\theta)$.*

Proof. Suppose that a point $y \neq x$ on $Imp(\theta)$ is separated from x in X by a finite set. By Proposition 4.1, we can find a good cut γ that separates x from y. By Lemma 6.3, we may assume that $\gamma \cap U_\infty$ is an open arc γ_1. Let a and b be the two end points of $\alpha_1 = \Phi^{-1}(\gamma_1)$, an open arc in $\mathbb{C} \setminus \overline{\mathbb{D}}$.

Fix an open arc $\mathcal{R}_\theta$ in $\mathbb{C}\setminus\overline{\mathbb{D}}$ landing at the point $e^{\mathbf{i}\theta}$ such that $\overline{\Phi(\mathcal{R}_\theta)} \cap X = Imp(\theta)$. We note that $e^{\mathbf{i}\theta} \in \{a, b\}$. Otherwise, there is a number $r > 1$ such that $\mathcal{R}_\theta \cap \{z : |z| < r\}$ lies in the component of

$$(\mathbb{C} \setminus \overline{\mathbb{D}}) \setminus (\{a, b\} \cup \alpha_1) \qquad \text{(difference of } \mathbb{C} \setminus \overline{\mathbb{D}} \text{ and } \{a, b\} \cup \alpha_1)$$

whose closure contains $e^{\mathbf{i}\theta}$. From this we see that $\Phi(\mathcal{R}_\theta \cap \{z : |z| < r\})$ is disjoint from γ and is entirely contained in one of the two components of $\mathbb{C} \setminus \gamma$, which contain x and y respectively. Therefore,

$$\overline{\Phi(\mathcal{R}_\theta \cap \{z : |z| < r\})}$$

hence its subset $Imp(\theta)$ cannot contain x and y at the same time. This contradicts the assumption that $x, y \in Imp(\theta)$.

Now we will lose no generality by assuming that $e^{\mathbf{i}\theta} = a$. Then $\Phi(\mathcal{R}_\theta)$ intersects γ_1 infinitely many times, since $\overline{\Phi(\mathcal{R}_\theta)} \setminus \Phi(\mathcal{R}_\theta)$ contains $\{x, y\}$. This implies that a is the landing point of $\mathcal{R}_\theta \subset (\mathbb{C} \setminus \overline{\mathbb{D}})$.

Let $w = \lim_{z\to a} \Phi|_{\alpha_1}(z)$. Then $\{x, y, w\} \subset Imp(\theta)$, and the proof will be completed as soon as we verify that $Imp(\theta) \subset F^*_w$.

Suppose there is a point $w_1 \in Imp(\theta)$ that is not in F^*_w. By Lemma 6.3 we may find a good cut γ' separating w from w_1 that intersects U_∞ at an open arc γ'_1. Let $\alpha'_1 = \Phi^{-1}(\gamma'_1)$. Since $w \notin \gamma'$, the closure $\overline{\alpha'_1}$ does not contain the point a. Let I be the component of $S^1 \setminus \overline{\alpha'_1}$ that contains a. Then $\mathcal{R}_\theta \cap \{z : |z| < r_1\}$ is disjoint from α'_1 for some $r_1 > 1$. For such an r_1, the image $\Phi(\mathcal{R}_\theta \cap \{z : |z| < r_1\})$ is disjoint from γ'. On the other hand, the good cut γ' separates w from w_1. Therefore, the closure of $\Phi(\mathcal{R}_\theta \cap \{z : |z| < r_1\})$ hence its subset $Imp(\theta)$ does not contain the two points w and w_1 at the same time. This is a contradiction. $\square$

Remark 6.5. Let J_c be the connected Julia set of a polynomial. The classes $[x]_\sim$ obtained in this paper determine a monotone decomposition of J_c, such that the quotient space is a locally connected continuum. Theorem 6.4 says that the impression of any prime end is entirely contained in a single class $[x]_\sim$. Therefore, the finest decomposition mentioned in [1, Theorem 1] is finer than $\{[x]_\sim : x \in J_c\}$. Currently it is not clear whether these two decompositions just coincide.

7. Facts and Examples

In this section, we give several examples to demonstrate the difference between (1) separating and cutting sets, (2) the modified fiber F^*_x and the class $[x]_\sim$, (3) a continuum $X \subset \mathbb{C}$ and the quotient space $X/\!\sim$ for specific choices of X. We also construct an infinite sequence of continua which have modified NLC-scales of any $k \geq 2$ and even up to ∞, although the quotient of each of those continua is always homeomorphic with the unit interval $[0, 1]$.

Example 7.1 (Separating Sets and Cutting sets). For a set $M \subset \mathbb{C}$, a set $C \subset M$ is said to *separate* or to be a *separating set between* two points $a, b \subset M$ if there is a separation $M \setminus C = P \cup Q$ satisfying $a \in P, b \in Q$; and a subset $C \subset M$ is called a *cutting set between two points* $a, b \in M$ if $\{a, b\} \subset (X \setminus C)$ and if the component of $X \setminus C$ containing a does not contain b [8, p.188,§47.VIII].

Let L_1 be the segment between the points $(2, 1)$ and $(2, 0)$ on the plane, Q_1 the one between $(-2, 0)$ and $c = (0, \frac{1}{2})$, and P_1 the broken line connecting

$(2, 0)$ to $(-2, 0)$ through $(0, -1)$, as shown in Figure 3. Define $(x_1, x_2) \xrightarrow{f}$

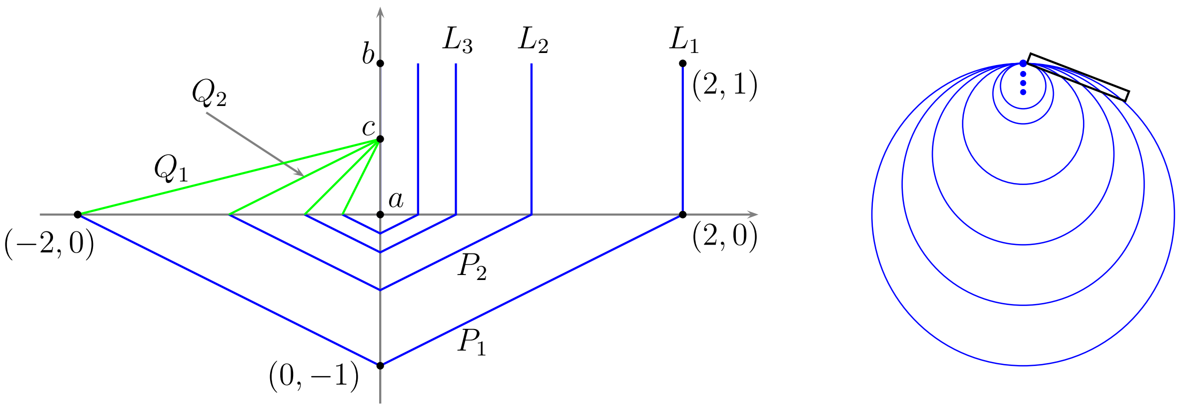

FIGURE 3. The continuum X and its quotient as a Hawaiian earring minus an open rectangle.

$(\frac{1}{2}x_1, \frac{1}{2}x_2)$ and $(x_1, x_2) \xrightarrow{g} (\frac{1}{2}x_1, x_2)$. For any $k \geq 1$, let $L_{k+1} = g(L_k)$ and $Q_{k+1} = g(Q_k)$; let $P_{k+1} = f(P_k)$. Let $B_k = L_k \cup P_k \cup Q_k$. Then $\{B_k : k \geq 1\}$ is a sequence of broken lines converging to the segment B between $a = (0, 0)$ and $b = (0, 1)$. Let $N = (\bigcup_k B_k) \bigcup B$. Then N is a continuum, which is not locally connected at each point of B. Moreover, the singleton $\{c\}$ is a cutting set, but not a separating set, between the points a and b. The only nontrivial modified fiber is $B = \{0\} \times [0, 1] = F_x^*$ for each $x \in B$. So we have $\ell^*(N) = 1$. Also, it follows that $[x]_\sim = B$ for all $x \in B$ and $[x]_\sim = \{x\}$ otherwise. In particular, the broken lines B_k are still arcs in the quotient space but, under the metric of quotient space, their diameters converge to zero. Consequently, the quotient $N/\sim$ is topologically the difference of a Hawaiian earring with a full open rectangle. See the right part of Figure 3. In other words, the quotient space $N/\sim$ is homeomorphic with the quotient $X/\sim$ of Example 7.2.

Example 7.2 (The Witch's Broom). Let X be the witch's broom [11, p.84, Figure 5.22]. See Figure 4. More precisely, let $A_0 := [\frac{1}{2}, 1] \times \{0\}$; let

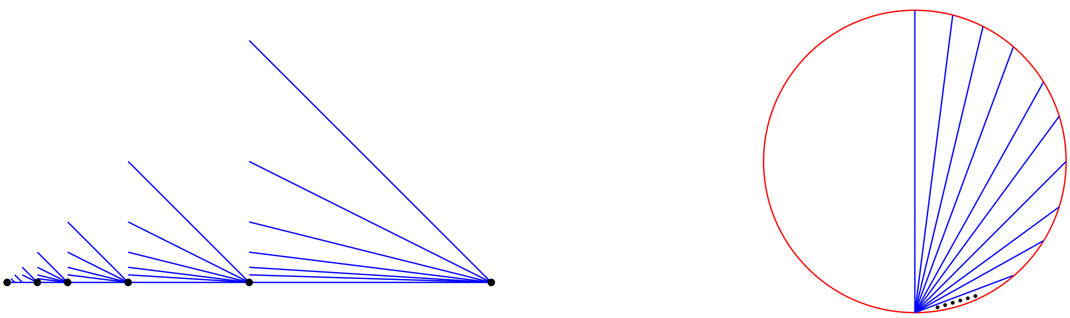

FIGURE 4. An intuitive depiction of the witch's broom and its quotient space.

A_k be the segment connecting $(1,0)$ to $(\frac{1}{2}, 2^{-k})$ for $k \geq 0$. Then $A = \bigcup_{k\geq 0} A_k$ is a continuum (an infinite broom) which is locally connected everywhere but at the points on $[\frac{1}{2}, 1) \times \{0\}$. Let $g(x) = \frac{1}{2}x$ be a similarity contraction on $\mathbb{R}^2$. Let

$$X = \{(0,0)\} \cup A \cup f(A) \cup f^2(A) \cup \cdots \cup f^n(A) \cup \cdots\cdots .$$

The continuum X is called the *Witch's Broom.* Consider the modified fibers of X, we have $F_x^* = \{x\}$ for each x in $X \cap \{(x_1, x_2) : x_2 > 0\}$ and for $x = (0,0)$. The nontrivial modified fibers include: $F_{(1,0)}^* = [\frac{1}{2}, 1] \times \{0\}$, $F_{(2^{-k},0)}^* = [2^{-k-1}, 2^{-k+1}] \times \{0\} \quad (k \geq 1)$, and

$$F_{(x_1,0)}^* = [2^{-k}, 2^{-k+1}] \times \{0\} \quad (2^{-k} < x_1 < 2^{-k+1}, k \geq 1).$$

Consequently, $[x]_\sim = \{x\}$ for each x in $X \cap \{(x_1, x_2) : x_2 > 0\}$, while $[x]_\sim = [0,1] \times \{0\}$ for $x \in [0,1] \times \{0\}$. See the right part of Figure 4 for a depiction of the quotient $X/\sim$.

Example 7.3 (**Witch's Double Broom).** Let X be the witch's broom. We call the union Y of X with a translated copy $X + (-1, 0)$ the *witch's double broom* (see Figure 5). Define $x \approx y$ if there exist points $x_1 = x$, $x_2, \ldots, x_n = y$ in Y such that $x_i \in F_{x_{i-1}}^*$. Then $\approx$ is an equivalence and is not closed. Its closure $\approx^*$ is not transitive, since we have $(-1,0) \approx^* (0,0)$ and $(0,0) \approx^* (1,0)$, but $(-1,0)$ is not related to $(1,0)$ under $\approx^*$.

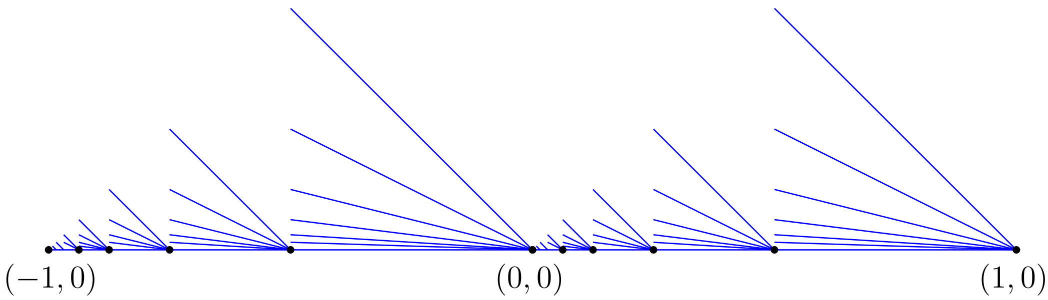

FIGURE 5. Relative locations of the points $(\pm 1, 0)$ and $(0,0)$ in witch's double broom.

Example 7.4 (**Cantor's Teepee).** Let X be Cantor's Teepee [14, p.145]. See Figure 6. Then the modified fiber $F_p^* = X$; and for every other point x, F_x^* is exactly the line segment on X that crosses x and p. Therefore, $\ell^*(X) = 1$. Moreover, $[x]_\sim = X$ for every x, hence the quotient is a single point. In this case, we also say that X *is collapsed to a point.*

Example 7.5 (**Cantor's Comb).** Let $\mathcal{K} \subset [0,1]$ be Cantor's ternary set. Let X be the union of $\mathcal{K} \times [0,1]$ with $[0,1] \times \{1\}$. See Figure 7. We call

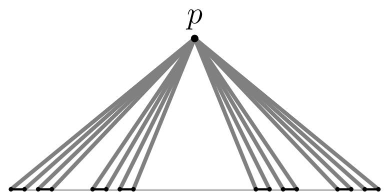

FIGURE 6. A simple representation of Cantor's Teepee.

X the Cantor comb. Then the modified fiber $F_x^* = \{x\}$ for every point on X that is off $\mathcal{K} \times [0,1]$; and for every point x on $\mathcal{K} \times [0,1]$, the modified fiber F_x^* is exactly the vertical line segment on $\mathcal{K} \times [0,1]$ that contains x. Therefore, $\ell^*(X) = 1$. Moreover, $[x]_\sim = F_x^*$ for every x, hence the quotient is homeomorphic to $[0,1]$. Here, we note that X is locally connected at every point lying on the common part of $[0,1] \times \{1\}$ and $\mathcal{K} \times [0,1]$, although the modified fibers at those points are each a non-degenerate segment.

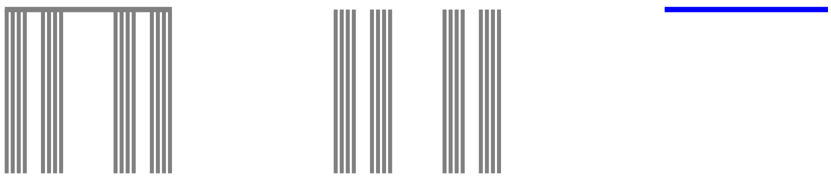

FIGURE 7. Cantor's Comb, its nontrivial modified fibers, and the quotient $X/\sim$.

Example 7.6 (More Combs). We use Cantor's ternary set $\mathcal{K} \subset [0,1]$ to construct a sequence of continua $\{X_k : k \geq 1\}$, such that the modified NLC-scale $\ell^*(X_k) = k$ for all $k \geq 1$. We also determine the modified fibers and compute the quotient spaces $X_k/\sim$. Let X_1 be the union of $X_1' = (\mathcal{K}+1) \times [0,2]$ with $[1,2] \times \{2\}$. Here $\mathcal{K}+1 := \{x_1 + 1 : x_1 \in \mathcal{K}\}$. Then X_1 is homeomorphic with Cantor's Comb defined in Example 7.5. We have $\ell^*(X_1) = 1$ and that $X_1/\sim$ is homeomorphic with $[0,1]$. Let X_2 be the union of X_1 with $[0,1] \times (\mathcal{K}+1)$. See Figure 8. Then the modified fiber of X_2 at the point $(1,2) \in X_2$ is $F^*_{(1,2)} = X_2 \cap \{(x_1,x_2) : x_1 \leq 1\}$, which will be referred to as the "largest modified fiber", since it is the modified fiber in X_2 that has the largest modified NLC-scale. See the central part of Figure 8. The other modified fibers are either a single point or a segment, of the form $\{(x_1,x_2) : 0 \leq x_2 \leq 2\}$ for some $x_1 \in \mathcal{K}+1$. Therefore, we have $\ell^*(X_2) = 2$ and can check that the quotient $X_2/\sim$ is homeomorphic with $[0,1]$. Let X_3 be the union of X_2 with

$$\frac{X_1}{2} = \left\{ \left(\frac{x_1}{2}, \frac{x_2}{2} \right) : (x_1,x_2) \in X_1 \right\}.$$

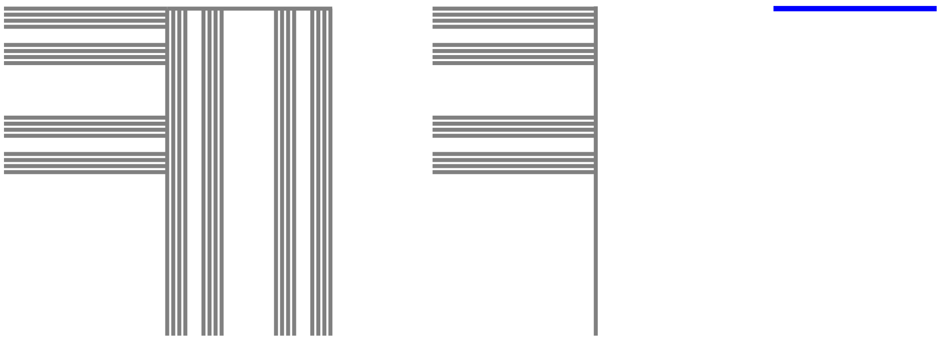

FIGURE 8. A simple depiction of X_2, its largest modified fiber, and the quotient $X_2/\sim$.

Then the largest modified fiber of X_3 is exactly $F^*_{(1,2)} = X_3 \cap \{(x_1, x_2) : x_1 \leq 1\}$, which is homeomorphic with X_2. Therefore, $\ell^*(X_3) = 3$; moreover, $X_3/\sim$ is also homeomorphic with $[0,1]$. See upper part of Figure 9. Let

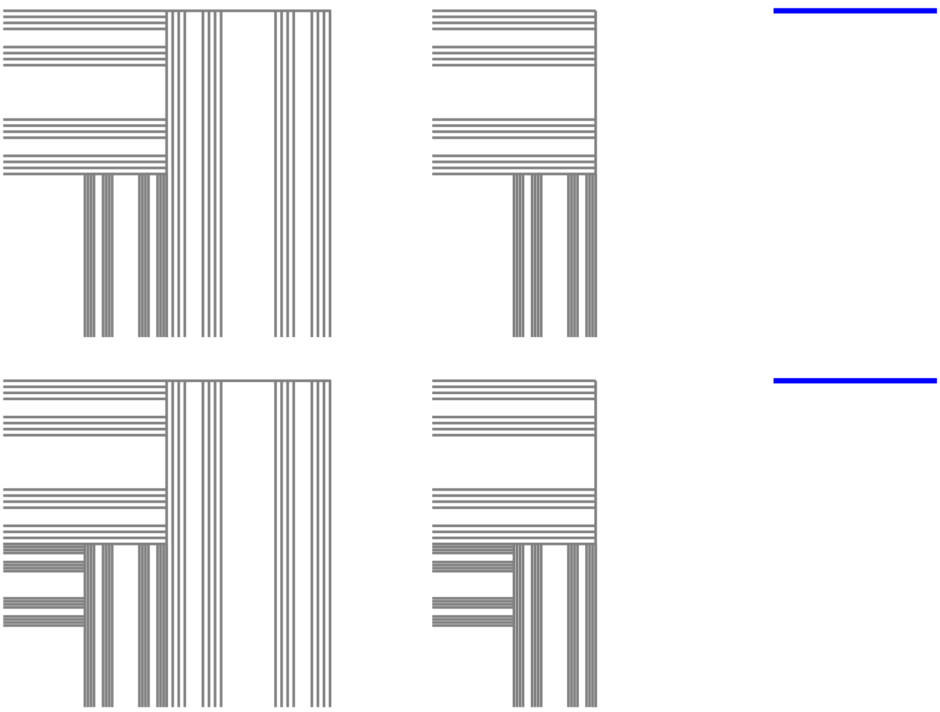

FIGURE 9. A depiction of X_3, X_4, their largest modified fibers, and the quotients $X_3/\sim$ and $X_4/\sim$.

$X_4 = X_2 \cup \frac{X_2}{2}$. Then the largest modified fiber of X_4 is $F^*_{(1,2)} = X_4 \cap \{(x_1, x_2) : x_1 \leq 1\}$, which is homeomorphic with X_3. Similarly, we can infer that $\ell^*(X_4) = 4$ and that $X_4/\sim$ is homeomorphic with $[0,1]$. See lower part of Figure 9. The construction of X_k for $k \geq 5$ can be done inductively. The general formula $X_{k+2} = X_2 \bigcup \frac{1}{2} X_k$ defines a path-connected continuum for all $k \geq 3$, for which the largest modified fiber is homeomorphic to X_{k+1}. Therefore, we have $\ell^*(X_k) = k$; moreover, the quotient space $X_k/\sim$ is always homeomorphic to the interval $[0,1]$. Finally, we can verify that

$$X_\infty = \{(0,0)\} \cup \left(\bigcup_{k=2}^{\infty} X_k \right)$$

is a path connected continuum and that its largest modified fiber is homeomorphic to X_∞ itself. Therefore, X_∞ has a modified NLC-scale $\ell^*(X_\infty) = \infty$, and the quotient $X_\infty/\sim$ is also homeomorphic to $[0,1]$.

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