A Core Decomposition of Compact Sets in the Plane

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Abstract

A compact metric space is called a *generalized Peano space* if all its components are locally connected and if for any constant $C > 0$ all but finitely many of the components are of diameter less than C. Given a compact set $K \subset \mathbb{C}$, there usually exist several upper semi-continuous decompositions of K into subcontinua such that the quotient space, equipped with the quotient topology, is a generalized Peano space. We show that one of these decompositions is finer than all the others and call it the *core decomposition of K with Peano quotient*. For specific choices of K, this core decomposition coincides with two models obtained recently, namely the locally connected models for unshielded planar continua (like connected Julia sets of polynomials) and the finitely Suslinian models for unshielded planar compact sets (like disconnected Julia sets of polynomials). We further answer several questions posed by Curry in 2010. In particular, we can exclude the existence of a rational function whose Julia set is connected and does not have a finest locally connected model.

Keywords. *Locally connected, finitely Suslinian, core decomposition.*

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1 Introduction and main results

In this paper, a compact metric space is called a *compactum* and a connected compactum is called a *continuum*. If K, L are two compacta, a continuous onto map $\pi : K \to L$ such that the preimage of every point in L is connected is called *monotone* [21].

We are interested in compacta and continua in the plane or in the Riemann sphere. Given a compactum $K \subset \mathbb{C}$, an *upper semi-continuous decomposition* $\mathcal{D}$ of K is a partition of K such that for every open set $B \subset K$ the union of all $D \in \mathcal{D}$ with $D \subset B$ is open in K (see [13]). Let π be the natural projection sending $x \in K$ to the unique element of $\mathcal{D}$ containing x. Then a set $A \subset \mathcal{D}$ is said to be open in $\mathcal{D}$ if and only if $\pi^{-1}(A)$ is open in K. This defines the *quotient topology* on $\mathcal{D}$. An upper semi-continuous decomposition of K is *monotone* if each of its elements is a subcontinuum of K. In this case, π is a monotone map. Finally, let $\mathcal{D}$ and $\mathcal{D}'$ be two monotone decompositions of a compactum $K \subset \mathbb{C}$, with projections π and π', and suppose that $\mathcal{D}$ and $\mathcal{D}'$ both satisfy a topological property (T). We say that $\mathcal{D}$ is *finer than* $\mathcal{D}'$ *with respect to* (T) if there is a map $g : \mathcal{D} \to \mathcal{D}'$ such that $\pi' = g \circ \pi$. If a monotone decomposition of a compactum K is finer than every other monotone decomposition of K with respect to (T), then it is called the *core decomposition* of K with respect to (T) and it will be denoted by $\mathcal{D}_K^T$ or simply $\mathcal{D}_K$, if (T) is fixed. Clearly, the core decomposition $\mathcal{D}_K$ is unique, if it exists.

Recently, core decompositions with respect to two specific topological properties were studied for the special class of compacta $K \subset \mathbb{C}$ that are *unshielded*, that is, $K = \partial W$ for the unbounded component W of $\mathbb{C} \setminus K$. More generally, when dealing with subsets of the Riemann sphere, a compactum $K \subset \hat{\mathbb{C}}$ is called unshielded if $K = \partial W$ for some component W of $\hat{\mathbb{C}} \setminus K$. In particular, if a rational function $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ has a completely invariant Fatou component, then its Julia set is an unshielded compactum (see [1, Theorem 5.2.1.(i)]).

First, Blokh-Curry-Oversteegen prove in [4, Theorem 1] that the core decomposition $\mathcal{D}_K^{LC}$ of an unshielded continuum K with respect to the property of being locally connected always exists. A special case is when K is the connected Julia set of a polynomial. In our paper, we will solve the existence of $\mathcal{D}_K^{LC}$ for all continua $K \subset \mathbb{C}$, without assuming that K is unshielded. In particular, our core decomposition will apply to the study of connected Julia sets of rational functions on the extended complex plane $\hat{\mathbb{C}}$, thus negatively answers [6, Question 5.2]. However, if we only consider upper semi-continuous decompositions then there might be two decompositions $\mathcal{D}_1, \mathcal{D}_2$ of an unshielded continuum $K \subset \mathbb{C}$ which are both Peano continua under quotient topology, such that the only decomposition finer than $\mathcal{D}_1$ and $\mathcal{D}_2$ is the decomposition $\{\{z\} : z \in K\}$ into singletons. Actually, let $\mathcal{C} \subset [0, 1] \subset \mathbb{C}$ be Cantor's ternary set. Let K be the union of

$\{x+\mathbf{i}y : x \in \mathcal{C}, y \in [0,1]\}$ with $\{x+\mathbf{i} : x \in [0,1]\}$ and call it the *Cantor Comb*. See the following figure for an approximation of K. Let p_1 be the restriction to K of the projection $x+\mathbf{i}y \mapsto x$.

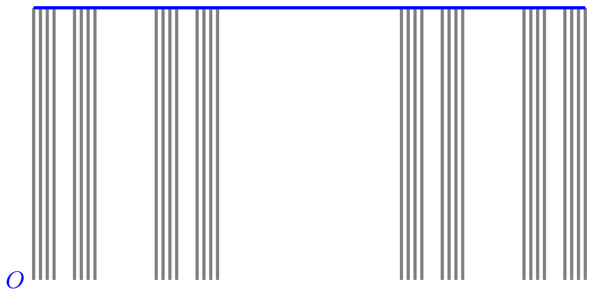

Figure 1: A rough approximation of the Cantor Comb.

Then $\mathcal{D}_1 = \left\{p_1^{-1}(x) : x \in [0,1]\right\}$ coincides with $\mathcal{D}_K^{LC}$, the core decomposition of K with respect to local connectedness. Let $\mathcal{D}_2$ be the union of all the translates $C_y := \{x+\mathbf{i}y : x \in \mathcal{C}\}$ of K with $0 \leq y \leq 1$ and all the single point sets $\{z = x+\mathbf{i}\}$ with $x \notin \mathcal{C}$. Then $\mathcal{D}_2$ is an upper semi-continuous decomposition of K, which is not monotone and which is a Hawaiian earing under quotient topology. Clearly, the only decomposition finer than both $\mathcal{D}_1$ and $\mathcal{D}_2$ is the decomposition $\{\{x\} : x \in K\}$ into singletons.

Second, given an unshielded compactum $K \subset \mathbb{C}$, the result of [2, Theorem 4] indicates the existence of the core decomposition of K with respect to the property of being finitely Suslinian, denoted by $\mathcal{D}_K^{FS}$. Here we recall that a compactum is *finitely Suslinian* if every collection of pairwise disjoint subcontinua whose diameters are bounded away from zero is finite. Since every finitely Suslinian continuum is locally connected, we see that [4, Theorem 1] is a special case of [2, Theorem 4]. We may wonder about the existence of the core decomposition $\mathcal{D}_K^{FS}$ of an arbitrary compactum $K \subset \mathbb{C}$. However, there are examples of continua $K \subset \mathbb{C}$ failing to have such a core decomposition (see [2, Example 14] and Section 2 of this paper). We will replace the property of being finitely Suslinian by the property of being a *generalized Peano space*. This class of compacta will be defined below. For every compactum $K \subset \mathbb{C}$, we will prove the existence of the core decomposition $\mathcal{D}_K^{PS}$ with respect to the property of being a generalized Peano space. We will briefly call $\mathcal{D}_K^{PS}$ the *core decomposition of K with Peano quotient*. Since a finitely Suslinian compactum is also a generalized Peano space, the decomposition $\mathcal{D}_K^{PS}$ is finer than $\mathcal{D}_K^{FS}$ for any compactum $K \subset \mathbb{C}$, when the latter exists. If the compactum $K \subset \mathbb{C}$ is unshielded, we will prove in Section 2 that our core decomposition $\mathcal{D}_K^{PS}$ coincides with the finest finitely Suslinian model $\mathcal{D}_K^{FS}$, given by [2, Theorem 4].

The core decomposition $\mathcal{D}_K^{LC}$ we will obtain just coincides with $\mathcal{D}_K^{PS}$, when $K \subset \mathbb{C}$ is a continuum. It is unknown whether such a core decomposition exists for a general continuum or compactum K, *e.g.*, when K can not embedded into the plane.

Our work is motivated by recent studies and possible applications in the field of complex dynamics, but we will rather focus on the topological part. The concept of generalized Peano space has its origin in an ancient result by Schönflies. It also has motivations from some recent works by Blokh, Oversteegen and their colleagues. We will show that this property can be used advantageously in discussing core decompositions, besides the properties of being locally connected or finitely Suslinian. Schönflies' result reads as follows.

Theorem. [13, p.515, §61, II, Theorem 10]. If K is a locally connected compactum in the plane and if the sequence $R_1, R_2, \dots$ of components of $\mathbb{C} \setminus K$ is infinite, then the sequence of their diameters converges to zero.

The above theorem gives a necessary condition for planar compacta to be locally connected. This condition is also necessary for planar compacta to be finitely Suslinian, as we will prove in Theorem 4.1. However, in both cases, the condition is not sufficient. For instance, Sierpinski's universal curve is not finitely Suslinian but its complement has infinitely many components whose diameters converge to zero. Also, the closed topologist's sine curve is not locally connected but its complement has a single component. This motivates us to introduce the following condition, which happens to be necessarily fulfilled by every planar compactum K, if K is assumed to be finitely Suslinian or locally connected.

Schönflies Condition. A compactum K in the plane fulfills the *Schönflies condition* if for the region U bounded by any two parallel lines L_1 and L_2, the **difference** $\overline{U} \setminus K$ has at most finitely many components intersecting both L_1 and L_2.

Theorem 1. *If a compactum K in the plane is locally connected or finitely Suslinian then it satisfies the Schönflies condition.*

We will prove an equivalent formulation of the Schönflies condition in Lemma 3.3: a compactum $K \subset \mathbb{C}$ satisfies the Schönflies condition if and only if for the region U bounded by any two parallel lines L_1 and L_2, the **intersection** $\overline{U} \cap K$ has at most finitely many components that intersect both L_1 and L_2. Moreover, we will show that the above Schönflies condition entirely characterizes the local connectedness for continua in the plane.

Theorem 2. *A continuum K in the plane is locally connected if and only if it satisfies the Schönflies condition.*

Continua like Sierpinski's universal curve indicate that the above theorem does not hold if we replace *locally connected* with *finitely Suslinian.*

The main purpose of this paper is not to characterize the finitely Suslinian compacta, but to find appropriate candidates for the core decomposition, instead of the finitely Suslinian property. Such a core decomposition of planar compacta will have interesting applications to the study on Julia sets of rational functions. See for instance the open questions proposed at the end of [6]. Combined with earlier models developed by Blokh-Curry-Oversteegen [2, 4], the results of Theorems 1 and 2 provide some evidence that planar compacta satisfying the Schönflies condition seem to be a reasonable model for the above mentioned core decompositions. Therefore, we give a nontrivial characterization of the Schönflies condition as follows.

Theorem 3. *A compactum $K \subset \mathbb{C}$ satisfies the Schönflies condition if and only if it has the following two properties.*

(1) Every component of K is locally connected.

(2) For every $C > 0$, all but finitely many components of K are of diameter less than C.

A compact metric space satisfying the above two properties in Theorem 3 will be called a *generalized Peano space*, simply *Peano space*. In particular, a Peano space is a Peano continuum if it is connected. Note that the term "Peano space" has been used as a synonym for Peano continuum in the literature (see [8, p.199] or [9, p.117]). In our paper, a connected Peano space means a Peano continuum, while a Peano space might be disconnected.

We are now in the position to introduce our strategy to prove the existence of $\mathcal{D}_K^{PS}$, the core decomposition with Peano quotient, for every compactum K in the plane.

Let us define a relation R_K on any given compactum $K \subset \mathbb{C}$ as follows. Given two disjoint simple closed curves J_1 and J_2, we denote by $U(J_1, J_2)$ the component of $\hat{\mathbb{C}} \setminus (J_1 \cup J_2)$ bounded by $J_1 \cup J_2$. This is an annulus in the extended complex plane $\hat{\mathbb{C}}$. We say that two points $x, y \in K$ are *related under* R_K provided that there exist two disjoint simple closed curves $J_1 \ni x$ and $J_2 \ni y$ such that $\overline{U(J_1, J_2)} \cap K$ contains an infinite sequence of components Q_k intersecting both J_1 and J_2, whose limit $\lim\limits_{k\to\infty} Q_k$ under Hausdorff distance contains $\{x, y\}$.

Definition 4. Let $K \subset \mathbb{C}$ be a compactum and R_K the relation defined above. We denote by $\mathcal{R}_K$ be the collection of all the closed equivalence relations on K containing R_K (as subsets of $K \times K$). Moreover, we denote by $\sim$ the intersection of all the equivalence relations of $\mathcal{R}_K$. It is also an element of $\mathcal{R}_K$, and we call it the *minimal equivalence containing* R_K, or the *Schönflies equivalence on* K, for short.

One can check that the equivalence class $[x]$ under the Schönflies equivalence $\sim$ is a continuum for every $x \in K$ (see Proposition 5.1). Moreover, if K in Theorem 3 is assumed to be unshielded, then it is finitely Suslinian if and only if it satisfies the Schönflies condition (see Theorem 2.1). Also, if the compactum K in Definition 4 is unshielded, then $\sim$ coincides with the relation developed in [2], given by the finest finitely Suslinian model (see Section 2).

From now on, we denote by $\mathcal{D}_K$ the collection of equivalence classes $[x] := \{z \in K : z \sim x\}$. Thus $\mathcal{D}_K$ is the decomposition induced by the Schönflies equivalence on K. It is standard to verify that $\mathcal{D}_K$ is necessarily a compact, Hausdorff and secondly countable space under quotient topology [11, p.148, Theorem 20]. Therefore, it is metrizable by Urysohn's metrization theorem [11, p.125, Theorem 16]. We will prove that it satisfies the two properties of Theorem 3, thus is a Peano space.

Theorem 5. *Under quotient topology $\mathcal{D}_K$ is a Peano space.*

After this we will show that $\mathcal{D}_K$ is finest in the following sense.

Theorem 6. *Let $\sim$ be the Schönflies equivalence on a compactum $K \subset \mathbb{C}$ and $\pi(x) = [x]$ the natural projection from K to $\mathcal{D}_K$. If $f : K \to Y$ is monotone map onto a Peano space Y, then there is an onto map $g : \mathcal{D}_K \to Y$ with $f = g \circ \pi$.*

By Theorems 5 and 6, we can conclude that $\mathcal{D}_K$ equals the core decomposition $\mathcal{D}_K^{PS}$ of K with Peano quotient.

Theorem 7. *Every compactum $K \subset \mathbb{C}$ has a core decomposition $\mathcal{D}_K^{PS}$ with respect to the property of being a Peano space. It coincides with the decomposition $\mathcal{D}_K$ induced by the Schönflies equivalence $\sim$ on K.*

Remark 8. Theorem 7 answers [6, Question 5.2] and partially answers [6, Question 5.2]. In the first part of [6, Question 5.2], Curry asks: for what useful topological properties P does there exist a finest decomposition of every Julia set $J(R)$ (of a rational function R) satisfying P? By Theorem 7, the property of "being a Peano space" is such a property. Moreover, in the last part of [6, Question 5.2], Curry asks: which of these (properties) is the appropriate analogue for the finest locally connected model? Since the core decomposition $\mathcal{D}_K^{PS}$ in Theorem 7 generalizes the earlier finest models obtained in [2, 4], the answer is again the property of "being a Peano space". Moreover, in the middle part of [6, Question 5.2], Curry asks: is the decomposition (satisfying the right property P) dynamic? This interesting question provides a reasonable angle to apply the core decomposition obtained in Theorem 7 to the study of complex dynamics, in particular, to the study on dynamics of a rational function restricted to its Julia set.

Remark 9. For any compactum K, the decomposition $\{Q : Q \text{ is a component of } K\}$ always induces a Peano quotient, whose components are single points. Therefore, an important problem is to determine whether the core decomposition $\mathcal{D}_K^{PS}$ of a given compactum $K \subset \mathbb{C}$ induces a quotient space having a non-degenerate component. In such a case, we say that $\mathcal{D}_K^{PS}$ is a *non-degenerate* core decomposition; otherwise, we say that it is a *degenerate* core decomposition. Clearly, the core decomposition of every indecomposable continuum $K \subset \mathbb{C}$ is degenerate, since $\mathcal{D}_K^{PS} = \{K\}$. The studies of Blokh-Curry-Oversteegen [2, 4], whose models are generalized by the core decomposition introduced in this paper, already provide very interesting results on the existence of non-degenerate core decompositions. For instance, by [4, Theorem 27], if a continuum $X \subset K$ has a "well-slicing family", then the image of X under the natural projection $\pi : K \to \mathcal{D}_K^{PS}$ is a non-degenerate continuum, hence K has a non-degenerate core decomposition. If K is the Julia set of a polynomial, then it is stated in [2, Corollary 24] that K has a non-degenerate core decomposition $\mathcal{D}_K^{PS}$ if and only if K has a periodic component Q which, as a plane continuum, has a non-degenerate core decomposition $\mathcal{D}_Q^{PS}$. In other words, to compute the core decomposition $\mathcal{D}_K^{PS}$ we just need to compute the core decomposition $\mathcal{D}_Q^{PS}$ for all the periodic components Q of K. If the above Julia set K is connected and is "finitely irreducible", the result of [6, Theorem 4.1] indicates that the core decomposition $\mathcal{D}_K^{PS}$ satisfies either $\mathcal{D}_K^{PS} = \{K\}$ or $\mathcal{D}_K^{PS} = \{\{x\} : x \in K\}$. Actually, in the former case K is an indecomposable continuum and in the latter case it is homeomorphic to $[0, 1]$. Finally, if X is an unshielded continuum and $Y \subset X$ is a subcontinuum, Blokh-Oversteegen-Timorin [5] obtained recently a sufficient condition for the core decomposition $\mathcal{D}_Y^{PS}$ of Y to embed canonically into that of X. As an application to complex dynamics, the authors also considered the special case that X is the connected Julia set of a renormalizable polynomial P and Y is the so-called *small Julia set*, for a polynomial-like map obtained as a restriction of some iterate P^n with $n > 1$. Combining these results with the core decomposition obtained in our paper, one may investigate problems like the local connectedness of Julia set of infinitely renormalizable polynomials.

We arrange our paper as follows. Section 2 briefly recalls facts on local connectedness, laminations in complex dynamics and core decompositions. We provide an argument based on Theorems 3 and 5 that the core decompositions $\mathcal{D}_K^{PS}$ and $\mathcal{D}_K^{FS}$ of an unshielded compactum $K \subset \mathbb{C}$ are equal. Section 3 gives preliminary lemmas needed in the proofs of the main theorems. Section 4 proves Theorems 1 to 3. Sections 5 and 6 respectively prove Theorems 5 and 6.

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2 Local Connectedness, Lamination, and Core Decomposition

The investigation of local connectedness dates back to the nineteenth century. Cantor proved that the unit interval and the unit square have the same cardinality. In other words, there exists a bijection $h : [0,1] \to [0,1]^2$, and this map h can not be continuous. Peano and some of his contemporaries further obtained continuous surjections from $[0,1]$ onto planar domains like squares and triangles. The range of a continuous map from $[0,1]$ into a metric space is therefore often called a *Peano continuum*. Peano continua were then fully characterized via the notion of local connectedness: indeed, Hahn and Mazurkiewicz showed that a continuum is a Peano continuum if and only if it is locally connected.

Among the Peano continua of the plane, the boundary of a bounded simply connected domain U provides a special case. By the Riemann Mapping Theorem, there is a conformal isomorphism from the unit open disk $\mathbb{D} = \{|z| < 1\}$ onto U. Furthermore, Carathéodory's theorem states that this conformal mapping has a continuous extension to the closed disk $\overline{\mathbb{D}}$ if and only if the boundary ∂U is locally connected. Considering U as a domain in the extended complex plane $\hat{\mathbb{C}}$, we may assume, after the action of a Möbius map, that $\infty \in U$. Then $X = \mathbb{C} \setminus U$ is a *full* continuum, *i.e.*, it has a connected complement U. Moreover, there is a conformal isomorphism Φ from $\mathbb{D}^* = \{|z| > 1\} \cup \{\infty\}$ onto U, fixing $\infty \in \hat{\mathbb{C}}$ and having a real derivative at ∞.

In the study of quadratic dynamics, examples of the above map Φ are (1) Böttcher maps for hyperbolic polynomials $z \mapsto z^2 + c$ with c lying in a hyperbolic component of the Mandelbrot set $\mathcal{M}$ and (2) the conformal isomorphism sending $\mathbb{D}^*$ onto $\hat{\mathbb{C}} \setminus \mathcal{M}$.

For the map Φ in (1), the boundary of $\Phi(\mathbb{D}^*)$ is the Julia set J_c of $z \mapsto z^2 + c$, which is known to be locally connected. In this case, J_c is the image of the unit circle $\partial\mathbb{D} = \partial\mathbb{D}^*$ under a continuous map (called Carathéodory's loop), hence may be considered as the quotient space of an equivalence relation on $\partial\mathbb{D}$. This equivalence relation is a *lamination* in Thurston's sense [20]. Douady [7] proposed a pinched disc model describing full locally connected continua in the plane. Extending the lamination in a natural way to a closed equivalence relation $\mathcal{L}$ on the closed unit disk $\overline{\mathbb{D}}$, he obtains that K_c is homeomorphic with the quotient $\overline{\mathbb{D}}/\mathcal{L}$, where K_c is the filled Julia set of the polynomial $z \mapsto z^2 + c$ ($J_c = \partial K_c$).

The pinched disc model works even if the full continuum K is not locally connected. The map Φ in (2) provides a typical example, in which the boundary of $\Phi(\mathbb{D}^*)$ coincides with that of the Mandelbrot set $\mathcal{M}$. Denote by $\mathcal{R}_\theta$ the image of $\{re^{2\pi\theta\mathbf{i}} : r > 1\}$ under Φ for $\theta \in [0,1]$. $\mathcal{R}_\theta$ is called the *external ray at* θ. If $\lim\limits_{r\to 1} \Phi\left(re^{2\pi\theta\mathbf{i}}\right)$ is a point on $\partial\mathcal{M}$, denoted as c_θ, we say that $\mathcal{R}_\theta$ *lands at* c_θ. It is known that all external rays $\mathcal{R}_\theta$ with rational θ lands. Douady therefore

[7] defines an equivalence relation on $\left\{e^{2\pi\theta\mathbf{i}}: \theta \in \mathbb{Q}\cap[0,1]\right\}$ by setting $\theta \sim_{\mathcal{M}}^{\mathbb{Q}} \theta'$ if and only if $c_\theta = c_{\theta'}$. As a subset of $\partial\mathbb{D}\times\partial\mathbb{D}$, the closure of $\sim_{\mathcal{M}}^{\mathbb{Q}}$ (denoted $\sim_{\mathcal{M}}$) turns out to be an equivalence relation on $\partial\mathbb{D}$ (see [7, Theorem 3] for fundamental properties of $\sim_{\mathcal{M}}$).

Let us now recall the main ideas of Blokh-Curry-Oversteegen [4] concerning locally connected models for unshielded continua in the plane. Let $K \subset \hat{\mathbb{C}}$ be an unshielded continuum with $K = \partial U$, where U is the unbounded component of $\mathbb{C}\setminus K$. Let Φ be a conformal mapping that sends $\mathbb{D}^*$ to U and fixes ∞. For any $\theta \in [0,1]$, the *impression* at $e^{2\pi\theta\mathbf{i}}$, defined by

$$\mathrm{Imp}(\theta) = \left\{\lim_{i\to\infty}\Phi(z_i): \ \{z_i\}\subset\mathbb{D}^*, \ \lim_{i\to\infty} z_i = e^{2\pi\theta\mathbf{i}}\right\},$$

is a subcontinuum of K. By [4, Lemma 13] there is a minimal closed equivalence relation $\mathcal{I}$ on K such that every equivalence class is made up of impressions and is a subcontinuum of K. By [4, Lemma 16], if R is an arbitrary closed equivalence on K such that the quotient space K/R is a locally connected continuum then $\mathcal{I}$ is contained in R (as subsets of $K\times K$). The first part of [4, Lemma 17] obtains that the quotient $K/\mathcal{I}$ is a locally connected continuum, called the locally connected model of K. Now we may define a closed equivalence relation $\sim_K$ on $\partial\mathbb{D}$ by requiring that $\theta \sim_K \theta'$ if and only if $\mathrm{Imp}(\theta)$ and $\mathrm{Imp}(\theta')$ lie in the same equivalence class $[x]_{\mathcal{I}}$. Then, by the second part of [4, Lemma 17], the equivalence $\sim_K$ is a lamination such that the induced quotient $\partial\mathbb{D}/\sim_K$ is homeomorphic to $K/\mathcal{I}$.

In particular, when K is the Julia set of a polynomial f of degree $d\geq 2$ without irrationally neutral cycles, Kiwi [12] investigates the structure of the classes $[x]_{\mathcal{I}}$ and shows that every $[x]_{\mathcal{I}}$ coincides with the *fiber at* $x\in K$ [12, Definition 2.5] defined by

$$\mathrm{Fiber}(x) = \{y \in K: \ \text{no finite set separates } y \text{ from } x \text{ in } K\}.$$

Here, a finite set $C\subset K$ *separates* two points of K if these points are in distinct components of $K\setminus C$. We refer to [12, Corollary 3.14] and [12, Proposition 3.15] for important properties of $\mathrm{Fiber}(x)$, and to Schleicher's earlier works [17, 18, 19] for another approach in defining fibers.

To generalize the above model, Blokh, Curry and Oversteegen [2] define an equivalence $\simeq$ on an unshielded compactum $K\subset\mathbb{C}$ to be the minimal closed equivalence such that every limit continuum is contained in a single class $[x]_\simeq := \{z\in K: z\simeq x\}$. Recall that a limit continuum is the limit $\lim\limits_{k\to\infty} N_k$ under Hausdorff distance of an infinite sequence of pairwise disjoint subcontinua $N_k\subset K$. The quotient space $\mathcal{D}_K^{FS} = \{[x]_\simeq : x\in K\}$ is necessarily a compact metrizable space [16, p.38, Theorem 3.9]. The authors of [2] further check that it is finitely Suslinian [2, Lemma 13].

Every element d of the above decomposition $\mathcal{D}_K^{FS}$, as a subset of $\mathbb{C}$, possesses the following property: the union of all the bounded components of $\mathbb{C} \setminus d$ does not intersect K. The authors of [2] then use Moore's theorem to prove that $\mathcal{D}_K^{FS}$ is the finest monotone decomposition of K with finitely Suslinian quotient [2, Theorem 19]. In other words, $\mathcal{D}_K^{FS}$ is the core decomposition of K with respect to the finitely Suslinian property.

Let now $K \subset \mathbb{C}$ be a any compactum. Let $\mathcal{D}_K^{\simeq} = \{[x]_{\simeq} : x \in K\}$ with $\simeq$ defined as above. On the other side, let $\sim$ be the Schönflies equivalence on K, defined in Definition 4 as the minimal closed equivalence relation containing the relation R_K. We write $\mathcal{D}_K = \{[x]_{\sim} : x \in K\}$. We want to compare these two decompositions. The definition of R_K indicates that if $(z_1, z_2) \in R_K$ then there is a limit continuum containing both z_1 and z_2. This in turn indicates that $\sim$ is contained in $\simeq$ as subsets of $K \times K$, hence the decomposition $\mathcal{D}_K$ always refines $\mathcal{D}_K^{\simeq}$.

These two decompositions turn out to be equal provide that K is unshielded. Note that in this case $\mathcal{D}_K^{\simeq} = \mathcal{D}_K^{FS}$ is the core decomposition of K with respect to the finitely Suslinian property [2, Theorem 19]. Actually, the unshielded assumption of K implies that the bounded components of $\mathbb{C} \setminus d$ for every $d \in \mathcal{D}_K$ are all disjoint from K. Let d^* be the union of d with the bounded components of $\mathbb{C} \setminus d$. Then

$$\mathcal{D}_{\mathbb{C}} := \{d^* : d \in \mathcal{D}_K\} \cup \left\{ \{z\} : z \notin \left(\bigcup_{d \in \mathcal{D}_K} d^* \right) \right\}$$

is a monotone decomposition of $\mathbb{C}$, such that $d_1^* \cap d_2^* = \emptyset$ for any $d_1 \neq d_2 \in \mathcal{D}_K$. By Moore's Theorem, the quotient $\mathcal{D}_{\mathbb{C}}$ is homeomorphic to the plane and the natural projection $\Pi : \mathbb{C} \to \mathcal{D}_{\mathbb{C}}$ sends K to a planar compactum. Since every d^* is disjoint from the unbounded component W of $\mathbb{C} \setminus K$, the image $\Pi(W)$ is a region in the plane $\mathcal{D}_{\mathbb{C}}$ whose boundary contains $\Pi(K)$. That is to say, $\Pi(K)$ is also an unshielded compactum in the plane $\mathcal{D}_{\mathbb{C}}$. On the other hand, for any $x, y \in K$ it is direct to check that $\Pi(x) = \Pi(y)$ if and only if $\pi(x) = \pi(y)$. Therefore, the quotient $\mathcal{D}_K$ is homeomorphic to $\Pi(K)$ hence may be embedded into the plane as an unshielded compactum. Theorem 5 of this paper says that $\mathcal{D}_K$ is also a Peano space. By the following theorem, such a planar compactum is finitely Suslinian. Consequently, the core decomposition decomposition $\mathcal{D}_K^{FS}$ is finer than $\mathcal{D}_K$, and we have $\mathcal{D}_K^{FS} = \mathcal{D}_K$.

Theorem 2.1. *If an unshielded compactum $K \subset \mathbb{C}$ is a Peano space then it is finitely Suslinian.*

Proof. By Theorem 3 and the definition of Peano space, we only need to consider the case when K is an unshielded continuum. Recall that a continuum X is *regular* at a point $x \in X$ if for every neighborhood V_x of x there exists a neighborhood U_x of x whose boundary $\partial U_x = \overline{U_x} \cap \overline{X \setminus U_x}$ is a finite set [22, p.19]. A *regular* continuum is just one that is regular at each of its points. Here it is standard to check that a regular continuum is finitely Suslinian. Therefore, our proof will be completed if only we can verify that K, a locally connected unshielded planar continuum, is a regular continuum.

We will use the notions of pseudo fiber and fiber for planar continua, recently introduced in [10], from which a numerical scale is developed that measures the extent to which such a continuum is locally connected.

More precisely, for any point $x \in K$, the pseudo fiber E_x at x consists of the points $y \in K$ such that there does not exist a simple closed curve γ with $\gamma \cap \partial X$ a finite set, called a good cut, such that x and y lie in different component of $\mathbb{C} \setminus \gamma$; the fiber F_x at x is the component of E_x that contains x. By [10, Proposition 4.2], for the locally connected unshielded continuum $K \subset \mathbb{C}$ every pseudo fiber E_x equals the single point set $\{x\}$. Therefore, given any $x \in K$ and any open set $U \ni x$, we can choose good cut γ_y such that x and y lie in different component of $\mathbb{C} \setminus \gamma_y$. Let U_y, V_y be the component of $\mathbb{C} \setminus \gamma_y$ with $x \in U_y$ and $y \in V_y$. Then $\{V_y : y \in K \setminus U\}$ is an open cover of the compact set $K \setminus U$. Fix a finite sub-cover $\{V_{y_1}, \ldots, V_{y_n}\}$. Then

$$U_x := \bigcap_{i=1}^{n} U_{y_i}$$

is open in K, contains x, and is contained in U. Recall that, for $1 \leq i \leq n$, the intersection $B_i := \overline{U_{y_i}} \cap \overline{V_{y_i}} \cap K$ is contained in $\gamma_{y_i} \cap K$ hence is also a finite set. Since the boundary of U_x in K is defined to be the intersection $\overline{U_x \cap K} \cap (K \setminus U_x)$ and is a subset of

$$\bigcup_{i=1}^{n} \left(\overline{U_x \cap K} \cap \overline{V_{y_i}} \cap K \right),$$

which is in turn a subset of $\bigcup_i B_i$ and hence is also a finite set. This verifies that K is regular at x. Consequently, from flexibility of $x \in K$ we can infer that K is a regular continuum. $\square$

Note that another proof of this theorem can be found in [3, Lemma 2.7].

We mention that if the compactum $K \subset \mathbb{C}$ is not assumed to be unshielded, then "the" core decomposition of K with respect to the finitely Suslinian property may not exist. Consider for instance the locally connected continuum $K \subset \mathbb{C}$ of [2, Example 14]. It admits two monotone decompositions $\mathcal{D}_1, \mathcal{D}_2$ such that the quotients are finitely Suslinian. However, the only partition finer than both $\mathcal{D}_1$ and $\mathcal{D}_2$ is the trivial decomposition $\{\{x\} : x \in K\}$. Therefore, "the" core decomposition of K with respect to the finitely Suslinian property does not exist, while the trivial decomposition $\{\{x\} : x \in K\}$ is the core decomposition of K with respect to the property of being a Peano space.

We end up this section with an example of a continuum $K \subset \mathbb{C}$ having two properties. Firstly, the core decomposition $\mathcal{D}_K$ has an element d such that at least two components of $\mathbb{C} \setminus d$ intersect K; secondly, the resulted quotient space $\mathcal{D}_K$ can not be embedded into the plane.

Example 2.2. Let the compactum $K \subset \mathbb{C}$ be the union of the closure of the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and the spiral curve $L = \left\{\left(1 + e^{-t}\right) e^{2\pi \mathbf{i} t} : t \geq 0\right\}$. By routine works one may check that the core decomposition of K with respect to the property of being a Peano space is exactly given by

$$\mathcal{D}_K = \{\{x\} : x \in (\mathbb{D} \cup L)\} \cup \{\partial \mathbb{D}\}.$$

Clearly, the quotient space is the one-point union of a sphere with a segment, thus can not be embedded into the plane. □

3 Some Useful Lemmas

The lemmas in this section give a couple of results that are used in latter sections. Lemma 3.1 is from [14, Lemma 2.1] and will be used in proving Lemmas 3.2 and 3.3.

Lemma 3.1. *Suppose that $A \subset [0,1) \times [0,1]$ and $B \subset (0,1] \times [0,1]$ are disjoint closed sets. Then there exists a path in $[0,1]^2 \setminus (A \cup B)$ starting from a point in $(0,1) \times \{0\}$ and leading to a point in $(0,1) \times \{1\}$.*

Before stating the next lemma, we recall the following definitions and facts. For $X \subset \mathbb{C}$, we say that $X = A \cup B$ $(A, B \neq \emptyset)$ is a *separation of* X if $\overline{A} \cap B = A \cap \overline{B} = \emptyset$.

Remember that, if x_0 is a point in X then the *component* of X containing x_0 is the maximal connected set $P \subset X$ with $x_0 \in P$. The *quasi-component* of X containing x_0 is defined to be the set

$$Q = \{y \in X : \text{ no separation } X = A \cup B \text{ exists such that } x \in A, y \in B\}.$$

Equivalently, the quasi-component containing a point $p \in X$ may be defined as the intersection of all closed-open subsets of X containing p. Any component is contained in a quasi-component, and quasi-components coincide with the components whenever X is compact [13].

If X is compact we denote by X^* be the union of X with all the bounded components of $\mathbb{C} \setminus X$. We call X^* the *topological hull of* X, following Blokh-Curry-Oversteegen [2].

Lemma 3.2. *Let $K \subset \mathbb{C}$ be compactum and $x_0 \notin K$. Then x_0 lies in the unbounded component of $\mathbb{C} \setminus K$ provided that it does not lie in the topological hull P^* for any component P of K.*

Proof. Fix a large enough circular disk D_r with radius $r > 0$ whose interior contains K. For each component P of K, since x_0 is assumed to be in the unbounded component of $\mathbb{C} \setminus P$, we may choose a path α_P disjoint from P which joins x_0 to a fixed point $x_1 \in \partial D_r$. Let δ be a number smaller than the distance $\text{dist}(P, \alpha_P) := \{|z_1 - z_2| : z_1 \in P, z_2 \in \alpha_P\}$. As P is also a quasi-component of K, we may choose a separation $K = A_{y,P} \cup B_{y,P}$ with $P \subset A_{y,P}$ and $y \in B_{y,P}$ for any point $y \in K$ with $\text{dist}(y, P) \geq \delta$. By compactness of $\{y \in K : \text{dist}(y, P) \geq \delta\}$, there are finitely many points $y_1, \ldots, y_l \in K$ with $\text{dist}(y_i, P) \geq \delta$ such that

$$B_{y_1,P}, B_{y_2,P}, \ldots, B_{y_l,P}$$

are open under the induced topology of K and form a cover of $\{y \in K : \text{dist}(y, P) \geq \delta\}$. Clearly, $K = A_P \cup B_P$ is also a separation with $\alpha_P \cap A_P = \emptyset$, where

$$A_P = \bigcap_{i=1}^{l} A_{y_i,P} \quad \text{and} \quad B_P = \bigcup_{i=1}^{l} B_{y_i,P}.$$

Since every A_P is both open and closed under the induced topology of K, we see that there are finitely many components $P_1, \ldots, P_m$ with

$$K = \left(\bigcup_{i=1}^{m} A_{P_i} \right).$$

Rename α_{P_i} as α_i for $1 \leq i \leq m$. Let $A_1 = A_{P_1}$. And, for $2 \leq i \leq m$, let

$$A_i = A_{P_i} \setminus \left(\bigcup_{j=1}^{i-1} A_j \right) = A_{P_i} \setminus \left(\bigcup_{j=1}^{i-1} A_{P_j} \right).$$

Then $A_1, \ldots, A_m$ are disjoint compact sets such that $K \subset \bigcup_{i=1}^{m} A_i$. Moreover, every α_i is an arc satisfying $\alpha_i \cap A_i = \emptyset$. To finish our proof, we will show that there is a path α disjoint from K that joins x_0 to x_1.

Recall that the paths $\alpha_1, \alpha_2 : [0, 1] \to \mathbb{C}$, with common initial point $\alpha_i(0) = x_0$ and common endpoint $\alpha_i(1) = x_1$, are homotopic relative to $\{0, 1\}$ under the straight line homotopy $F : [0, 1]^2 \to \mathbb{C}$ defined by $F(t, s) = t\alpha_1(s) + (1 - t)\alpha_2(s)$ for any $t, s \in [0, 1]$. The disjointness of A_1, A_2 indicates that $F^{-1}(A_1), F^{-1}(A_2)$ are disjoint compact subsets of the unit square. By Lemma 3.1 there is a path β in $[0, 1]^2 \setminus (F^{-1}(A_1) \cup F^{-1}(A_2))$ starting from a point in $\{0\} \times (0, 1)$ and leading to a point in $\{1\} \times (0, 1)$. Then $F(\beta)$ is a path disjoint from $A_1 \cup A_2$ and joins x_0 to x_1.

Repeating the above argument on the paths $F(\beta)$ and α_3, we can find a path disjoint from $A_1 \cup A_2 \cup A_3$ that joins x_0 to x_1. Inductively, we can find a path α disjoint from $\cup_{i=1}^{m} A_i$, hence disjoint from K, that joins x_0 to x_1. □

Lemma 3.3. *Let $K \subset \mathbb{C}$ be a compact set and U the region bounded by two parallel lines L_1 and L_2. Let U_i be the component of $\mathbb{C} \setminus (L_1 \cup L_2)$ with $\partial U_i = L_i$. We have the following:*

(1) If $K \cap L_1 \neq \emptyset \neq K \cap L_2$ and no connected subset of K intersects both L_1 and L_2, then there is a separation $K = A_1 \cup A_2$ with A_1, A_2 compact sets such that $(\overline{U_i} \cap K) \subset A_i$.

(2) If $\overline{U} \setminus K$ has at least $m \geq 2$ components intersecting both L_1 and L_2, then $\overline{U} \cap K$ has at least $m-1$ components intersecting both L_1 and L_2.

(3) If $\overline{U} \cap K$ has at least $m \geq 2$ components intersecting both L_1 and L_2, then $\overline{U} \setminus K$ has at least m components intersecting both L_1 and L_2.

Remark 3.4. The results of Lemma 3.3 still hold if we replace $\mathbb{C}$ by $\hat{\mathbb{C}}$, L_1, L_2 by two disjoint simple closed curves $J_1, J_2 \subset \hat{\mathbb{C}}$ and U by the region W bounded by $J_1 \cup J_2$. For Part (1), the argument is still valid. For Part (2), we may remove an open arc $\alpha \subset W$, which joins a point in J_1 to a point in J_2 and has a closure disjoint from K; then the difference $\overline{W} \setminus \overline{\alpha}$ contains $\overline{W} \cap K$ and has the same topology of $\overline{U}$. For part (3), one may use the pattern of polar brick wall tiling, as indicated in the figure below, to find such an open arc α.

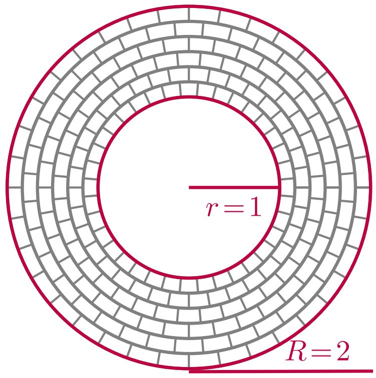

Figure 2: A Polar Brick Wall Tiling, in which every two tiles either are disjoint or intersect at a non-degenerate arc on the boundary.

Proof for Lemma 3.3. We first prove Item (1). Every component of the compact set $\overline{U} \cap K$ is also a quasi-component. Therefore, for any $a \in (L_1 \cap K)$ and any $b \in (L_2 \cap K)$, there exists a separation $\overline{U} \cap K = W_{a,b} \cup V_{a,b}$ with $a \in W_{a,b}$ and $b \in V_{a,b}$ such that the sets $W_{a,b}, V_{a,b}$ are closed in $\mathbb{C}$ and relatively open in $\overline{U} \cap K$. In particular, the collection $\{V_{a,b} : b \in (L_2 \cap K)\}$ is an open cover of the compact set $L_2 \cap K$ in $\overline{U} \cap K$, so there exists a finite subcover $V_{a,b_1}, ..., V_{a,b_k}$. Let

$$W_a = \bigcap_{i=1}^{k} W_{a,b_i} \quad \text{and} \quad V_a = \bigcup_{i=1}^{k} V_{a,b_i}.$$

Then $\overline{U} \cap K = W_a \cup V_a$ is a separation such that $a \in W_a$ and $(L_2 \cap K) \subset V_a$. By flexibility of $a \in L_1 \cap K$, the collection $\{W_a : a \in (L_1 \cap K)\}$ is an open cover of $L_1 \cap K$ in $\overline{U} \cap K$ and has a finite subcover $W_{a_1}, \ldots, W_{a_l}$. Let

$$W = \bigcup_{i=1}^{l} W_{a_i} \quad \text{and} \quad V = \bigcap_{i=1}^{l} V_{a_i}.$$

Then $\overline{U} \cap K = W \cup V$ is a separation such that $(L_1 \cap K) \subset W$ and $(L_2 \cap K) \subset V$. If we set $A_1 = W \cup (U_1 \cap K)$ and $A_2 = V \cup (U_2 \cap K)$, then $K = A_1 \cup A_2$ is a separation with $(\overline{U_1} \cap K) \subset A_1$ and $(\overline{U_2} \cap K) \subset A_2$. By construction, A_1 and A_2 are closed subsets of $\mathbb{C}$. This proves Item (1).

The proof of Item (2) reads as follows. Let $R_1, \ldots, R_m$ be m components of $\overline{U} \setminus K$ intersecting both L_1 and L_2. For each $i \in \{1, \ldots, m\}$, let α_i be a simple arc in the component R_i of $\overline{U} \setminus K$ with endpoints $a_i \in L_1, b_i \in L_2$ and $\alpha_i \setminus \{a_i, b_i\} \subset U \cap R_i$. Choose a line L perpendicular to L_1 such that all of the arcs $\{\alpha_i\}_{1 \leqslant i \leqslant m}$ are in the same component of $\mathbb{C} \setminus L$. We can assume that a_1 is the nearest point to L among the collection $\{a_i\}_{1 \leqslant i \leqslant m}$, and a_j is the nearest point to L among the set of points $\{a_i\}_{j \leqslant i \leqslant m}$, then the arcs α_i are renamed according to the distance of their endpoints on L_1 to the line L. Since the arcs α_i are disjoint, we can use Jordan curve theorem to infer that the distance from b_j to L is smaller than that from b_{j+1} to L for $1 \leq j \leq m-1$.

Let β_i be the arc on L_1 with endpoints a_i, a_{i+1} for $1 \leq i \leq m-1$. Let γ_i be the arc on L_2 with endpoints b_i, b_{i+1} for $1 \leq i \leq m-1$. Then $\Gamma_i = \alpha_i \cup \beta_i \cup \gamma_i \cup \alpha_{i+1}$ is a simple closed curve for $1 \leq i \leq m-1$. Let $W_i \subset U$ be the bounded component of $\mathbb{C} \setminus \Gamma_i$. By a theorem of Schönflies [15, p.68,Theorem 6], there is a homeomorphism $h_i : \overline{W_i} \to [0,1]^2$ such that

$$h_i(a_i) = (0,0), \quad h_i(a_{i+1}) = (0,1), \quad h_i(b_i) = (1,0), \quad h_i(b_{i+1}) = (1,1).$$

Since α_i and α_{i+1} lie in distinct components of $\overline{U} \setminus K$, the compact set $h_i\left(\overline{W_i} \cap K\right)$ intersects each of $h_i(\beta_i)$ and $h_i(\gamma_i)$ and is disjoint from $h_i(\alpha_i \cup \alpha_{i+1}) = \{0,1\} \times [0,1]$. See left part of Figure 3 for relative locations of $h_i(\alpha_i)$, $h_i(\alpha_{i+1})$, $h_i(\beta_i)$ and $h_i(\gamma_i)$.

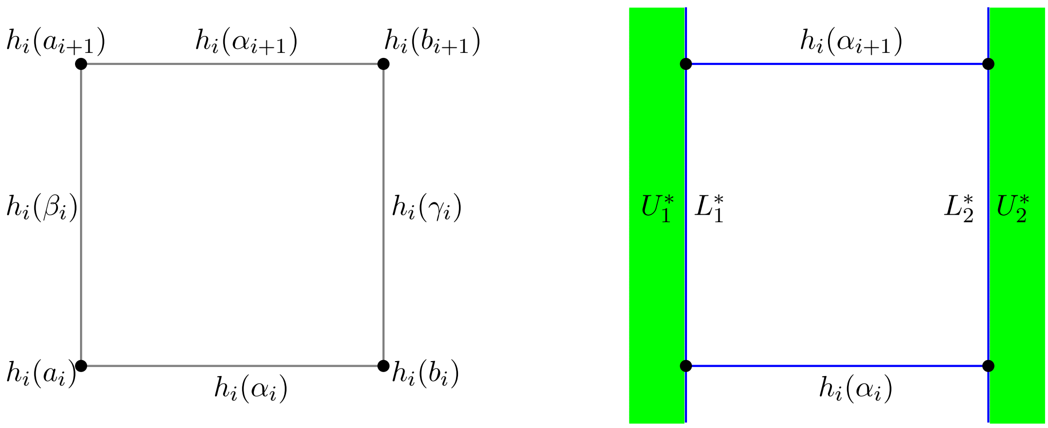

Figure 3: Relative locations of the points $h_i(a_i), h_i(b_i), h_i(a_{i+1}), h_i(b_{i+1})$ in $[0,1]^2$.

We claim that $h_i\left(\overline{W_i} \cap K\right)$ has a component P_i that intersects both $h_i(\beta_i)$ and $h(\gamma_i)$. This will verify Item (2), since P_i is a closed subset of $[0,1]^2$, hence $N_i := h_i^{-1}(P_i) \subset \left(\overline{W_i} \cap K\right)$ is a sub-continuum of K and intersects both L_1 and L_2. Note that $N_i \cap N_{i+1} = \emptyset$, since $\overline{W_i} \cap \overline{W_{i+1}} \cap K = \alpha_{i+1} \cap K = \emptyset$.

Suppose on the contrary that $h_i\left(\overline{W_i}\cap K\right)$ has no component intersecting both $h_i(\beta_i)$ and $h(\gamma_i)$. We will use Item (1) to induce a contradiction.

To this end, we put $K^* = h_i\left(\overline{W_i}\cap K\right)$. Then, let L_1^* be the line through $h_i(a_i), h_i(a_{i+1})$ and L_2^* the line through $h_i(b_i), h_i(b_{i+1})$. Moreover, let U_i^* be the component of $\mathbb{C}\setminus(L_1^*\cup L_2^*)$ with $\partial U_i^* = L_i^*$. See right part of Figure 3.

By Item (1), we are able to find a separation $K^* = A_1^*\cup A_2^*$ into compact sets with $(\overline{U_i^*}\cap K^*)\subset A_i^*$. It follows that $A_1^*\subset[0,1)\times[0,1]$ and $A_2^*\subset(0,1]\times[0,1]$. In particular, A_1^* and A_2^* satisfy the conditions in Lemma 3.1, from which we can infer the existence of a path P in $[0,1]^2\setminus(A_1^*\cup A_2^*)$ starting at a point in $(0,1)\times\{0\}\subset h_i(\alpha_i)$ and leading to a point in $(0,1)\times\{1\}\subset h_i(\alpha_{i+1})$. The inverse $h_i^{-1}(P)$ is then a path in $\overline{W_i}\setminus K$ $(\subset\overline{U}\setminus K)$, which connects a point in $\alpha_i\subset R_i$ to a point in $\alpha_{i+1}\subset R_{i+1}$. This is impossible, since R_i, R_{i+1} are distinct components of $\overline{U}\setminus K$.

Finally we prove part (3). Let $Q_1,\ldots,Q_m$ be m components of $\overline{U}\cap K$ intersecting both L_1 and L_2. Clearly $\epsilon = \frac{1}{3}\min\{\mathrm{dist}(Q_i,Q_j) : 1\le i<j\le m\}$ is a positive number. For $1\le i\le m$, let $Q_i(\epsilon)$ be the open ϵ-neighborhood of Q_i. Then for every point x in $(\overline{U}\cap K)\setminus Q_i(\epsilon)$ the quasi-component of $\overline{U}\cap K$ containing x is disjoint from the one containing Q_i, so that there is a separation $\overline{U}\cap K = C_x\cup D_x$ with $Q_i\subset C_x$ and $x\in D_x$. Under the induced topology of $\overline{U}\cap K$, the collection

$$\left\{D_x : x\in(\overline{U}\cap K)\setminus Q_i(\epsilon)\right\}$$

is an open cover of the compact set $(\overline{U}\cap K)\setminus Q_i(\epsilon)$, which has a finite sub-cover

$$\left\{D_{x_k} : x_1,\cdots,x_n\in(\overline{U}\cap K)\setminus Q_i(\epsilon)\right\}.$$

Then $\overline{U}\cap K = C_i\cup D_i$ is a separation with $Q_i\subset C_i\subset Q_i(\epsilon)$ for $1\le i\le m$, where

$$C_i = \bigcap_{k=1}^{n} C_{x_k} \quad\text{and}\quad D_i = \bigcup_{k=1}^{n} D_{x_k}.$$

Moreover, $\overline{U}\cap K = C\cup D$ is also a separation, where

$$C = \bigcup_{i=1}^{m} C_i \quad\text{and}\quad D = \bigcap_{i=1}^{m} D_i.$$

Denote by d the distance between L_1 and L_2. Let $\delta = \min\{\mathrm{dist}(C,D),\epsilon\}$. Fix an integer $N_1>0$ with $\frac{d}{2N_1}<\frac{\delta}{4}$ and divide $\overline{U}$ into $2N_1$ equal strips by $2N_1-1$ lines parallel to L_1 such that the width of each strip is $\frac{d}{2N_1}$.

In the rest part of our proof we assume that L_1 is the horizontal axis and L_2 is on the upper half plane.

Let $\mathcal{T}_1 = \{B_k : k \in \mathbb{Z}\}$ be a tiling of the lowest strip by squares of side length $\frac{d}{2N_1}$, *such that $\mathcal{T}_1$ is a cover and the squares have disjoint interiors.* Then

$$\mathcal{T}_{2n-1} := \left\{ B_k + \left(\frac{(2n-2)d}{4N_1}, 0 \right) : k \in \mathbb{Z} \right\}$$

is a tiling of the $(2n-1)$-th strip for $2 \leq n \leq N_1$, and

$$\mathcal{T}_{2n} := \left\{ B_k + \left(\frac{(2n-1)d}{4N_1}, \frac{d}{4N_1} \right) : k \in \mathbb{Z} \right\}$$

is a tiling of the $(2n)$-th strip for $1 \leq n \leq N_1$. Moreover, $\mathcal{T} := \bigcup_{i=1}^{2N_1} \mathcal{T}_i$ is a tiling of the whole strip $\overline{U}$ that represents a "brick wall pattern", such that two squares either are disjoint or intersect at a non-degenerate segment.

For $1 \leq i \leq m$, let C_i' be the union of all the squares in $\mathcal{T}$ intersecting C_i. Let R_i be the component of C_i' containing Q_i. Clearly, we have $Q_i \subset C_i \subset C_i' \subset Q_i(\epsilon)$. By Torhorst Theorem [13, p.512, §61, II, Theorem 4], the unbounded component of $\mathbb{C} \setminus R_i$ is bounded by a simple closed curve J_i. Clearly, the curves $J_1, \ldots, J_m$ are pairwise disjoint.

Choose $a_i \in (J_i \cap L_1)$ and $b_i \in (J_i \cap L_2)$ with $\text{dist}(a_i, L) = \text{dist}(J_i \cap L_1, L)$ and $\text{dist}(b_i, L) = \text{dist}(J_i \cap L_2, L)$. Then $J_i \setminus \{a_i, b_i\}$ is made up of two open arcs; one of them must be contained in U and will be denoted as A_i. Let H be the union of L_1, L_2 and all the arcs $A_1, \ldots, A_m$. Fix a permutation $i_1 i_2 \cdots i_m$ of $1, 2, \ldots, m$ such that the distance from a_{i_k} to L is smaller than that from $a_{i_{k+1}}$ to L for $1 \leq k \leq m-1$. Then $\mathbb{C} \setminus H$ has exactly $m-1$ components $W_1, W_2, \ldots, W_{m-1}$ and the boundary of W_k is $A_{i_k} \cup A_{i_{k+1}} \cup \alpha_k \cup \beta_k$ for $1 \leq k \leq m$, where $\alpha_k \subset L_1$ and $\beta_k \subset L_2$ are minimal arcs containing $\{a_{i_k}, a_{i_{k+1}}\}$ and $\{b_{i_k}, b_{i_{k+1}}\}$, respectively.

Since $J_{i_k} \cap J_{i_{k+1}} = \emptyset$, from the choice of the points a_{i_k}, b_{i_k} and the arc A_{i_k}, we can infer that $J_{i_k} \setminus A_{i_k}$ is contained in the closure $\overline{W_{i_k}}$, which necessarily contains Q_{i_k}. Since Q_{i_k} intersects both L_1 and L_2, the arc A_{i_k} is separated from all the arcs A_{i_l} with $l > k$ by Q_{i_k} in $\overline{U}$. That is to say, all the arcs $A_1, A_2, \ldots, A_m$ lie in different components of $\overline{U} \setminus K$, indicating that $\overline{U} \setminus K$ has m components which intersect both L_1 and L_2. $\square$

Let R_K be the closed relation on a compact set $K \subset \mathbb{C}$, firstly mentioned before Definition 4. The following lemma provides an equivalent approach to define R_K. In the sequel, the bounded component of a simple closed curve $\Gamma \subset \mathbb{C}$ is denoted as $Int(\Gamma)$ and called the *interior of* Γ.

Lemma 3.5. *Given a compact set $K \subset \mathbb{C}$ and $n \geq 3$ disjoint simple closed curves $\Gamma_1, \ldots, \Gamma_n \subset \mathbb{C}$ such that $\Gamma_2, \ldots, \Gamma_n \subset Int(\Gamma_1)$ and $Int(\Gamma_j) \cap Int(\Gamma_j) = \emptyset$ for $i \neq j \geq 2$. Let W be the annulus bounded by Γ_1, Γ_2. Let W^* be the only region with $\partial W^* = \bigcup_{k=1}^{n} \Gamma_k$. If the intersection*

$\overline{W^*} \cap K$ *has infinitely many components* P_k *each of which intersects both* Γ_1 *and* Γ_2*, such that* $\lim_k P_k = P_\infty$ *under Hausdorff distance, then* $(z_1, z_2) \in R_K$ *for any* $z_1 \in (\Gamma_1 \cap P_\infty)$ *and* $z_2 \in (\Gamma_2 \cap P_\infty)$*.*

Proof. We only consider the case $n = 3$, to which the other cases may be reduced.

Fix a circular disk D whose interior contains K. By Lemma 3.2, there are two mutually exclusive possibilities: (1) the interior $Int(\Gamma_3)$ is contained in P^* for some component P of $\overline{W^*} \cap K$ or (2) a point $x_0 \in \Gamma_3$ may be connected to a point x_1 on the circle ∂D by a path α disjoint from $\overline{W^*} \cap K$.

In the former case, every P_k other than P is disjoint from P^*; each of those components is a component of $\overline{W} \cap (K \cup P^*)$ hence also a component of $\overline{W} \cap K$. This guarantees that $(z_1, z_2) \in R_K$ for any $z_1 \in (\Gamma_1 \cap P_\infty)$ and $z_2 \in (\Gamma_2 \cap P_\infty)$.

In the latter case, let y be the first point on α that also lies in $\Gamma_1 \cup \Gamma_2$ and $\alpha_0 \subset \alpha$ the irreducible sub-path with end points x_0, y. There are two subcases, $y \in \Gamma_2$ or $y \in \Gamma_1$, and we just consider the subcase $y \in \Gamma_2$ since the same argument applies to the other subcase.

We may slightly thicken α_0 and find two disjoint arcs α', α'' close enough to α_0, each of which does not intersect $\overline{W^*} \cap K$ and joins a point on Γ_3 to a point on Γ_2, such that α', α'' are contained in W^* except for their end points. Then the unbounded component of $\mathbb{C} \setminus (\Gamma_2 \cup \Gamma_3 \cup \alpha' \cup \alpha'')$ is bounded by a simple closed curve Γ_2' contained in $Int(\Gamma_2)$, such that the region W' bounded by Γ_1, Γ_2' is an annulus. Clearly, we have $\overline{W^*} \cap K = \overline{W'} \cap K$, and thus every P_k is also a component of $\overline{W'} \cap K$. Consequently, we have $(z_1, z_2) \in R_K$ for any $z_1 \in (\Gamma_1 \cap P_\infty)$ and $z_2 \in (\Gamma_2 \cap P_\infty) \subset (\Gamma_2' \cap P_\infty)$. $\square$

4 Proofs for Theorems 1 to 3

Firstly, we copy the ideas of Schönflies result [13, p.515, §61, II, Theorem 10] and obtain a necessary condition for a planar compactum to be finitely Suslinian.

Theorem 4.1. *Given a finitely Suslinian compactum* $K \subset \mathbb{C}$*. If the sequence* $R_1, R_2, \ldots$ *of components of* $\mathbb{C} \setminus K$ *is infinite then the sequence of their diameters converges to zero.*

Proof. Suppose conversely that there exists $\epsilon > 0$ and infinitely many integers $i_1 < i_2 < \cdots$ such that the diameter $\delta(R_{i_n}) > 3\epsilon$. For each component R_{i_n}, choose an arc $\alpha_{i_n} \subset R_{i_n}$ with diameter larger than 3ϵ. We may assume that α_{i_n} converges to α_0 under Hausdorff distance.

Here we have $\delta(\alpha_0) \geq 3\epsilon$. Choose two points $p', q' \in \alpha_0$ with $|p' - q'| = 3\epsilon$. Then, we can fix two points p_1, p_2 in the interior of the segment $\overline{pq}$ with $|p_1 - p_2| = 2\epsilon$.

Let L_i be the line through p_i which is perpendicular to $\overline{pq}$. Let U be the region bounded by $L_1 \cup L_2$. Since $\lim\limits_{n\to\infty} \alpha_{i_n} = \alpha_0$, there exists an integer N such that α_{i_n} intersects both L_1 and L_2 for all $n > N$. By part (2) of Lemma 3.3, there exist infinitely many components of $\overline{U} \cap K$ which intersect both L_1 and L_2. This contradicts the condition that K is finitely Suslinian. $\square$

Secondly, we prove Theorem 1 as follows.

Proof for Theorem 1. The part for locally connected compacta is a direct corollary of Schönflies' result [13, p.515, §61, II, Theorem 10]. So we only consider the part for finitely Suslinian compacta.

Suppose on the contrary that there exist two parallel lines L_1, L_2 such that the difference $\overline{U} \setminus K$ has infinitely many components $R_1, R_2, \ldots$ intersecting each of L_1 and L_2. Here U is the only component of $\mathbb{C} \setminus (L_1 \cup L_2)$ bounded by $L_1 \cup L_2$. By part (2) of Lemma 3.3, $\overline{U} \cap K$ has infinitely many components intersecting each of L_1 and L_2. This contradicts the assumption that K is finitely Suslinian. $\square$

Then we continue to prove Theorem 2.

Proof for Theorem 2. We just show that a continuum $K \subset \mathbb{C}$ satisfying the Schönflies condition is locally connected. Suppose on the contrary that K is not locally connected at a point $x_0 \in K$. By definition of local connectedness [13, p.227, §49, I, Definition], there would exist a **closed square** V centered at x_0 such that the component P_0 of $V \cap K$ containing x_0 is not a neighborhood of x_0 with respect to the induced topology on $V \cap K$. In other words, there exists a sequence $\{x_k\}_{k=1}^{\infty}$ in $(V \cap K) \setminus P_0$ with $\lim\limits_{k\to\infty} x_k = x_0$ such that the components of $V \cap K$ containing x_k, denoted P_k, are pairwise disjoint.

Recall that the hyperspace of all closed nonempty subsets of V is a compact metric space under Hausdorff distance. Coming to an appropriate subsequence, if necessary, we may assume that P_k converges to P_∞ in Hausdorff distance. From this, we see that P_∞ is a sub-continuum of P_0 and that the diameter of P_k, denoted $\delta(P_k)$, converges to $\delta(P_\infty)$.

By connectedness of K, each P_k must intersect ∂V hence P_∞ intersects ∂V. Since $P_k \to P_\infty$ under Hausdorff distance, we can pick some point $y_0 \in (\partial V \cap P_\infty)$ and points $y_k \in (\partial V \cap P_k)$ for all $k \geq 1$ such that $y_0 := \lim\limits_{k\to\infty} y_k$.

Since ∂V consists of four segments and contains the infinite set of points $\{y_k\}$,we may fix a line L_1 crossing infinitely many y_k, which necessarily contains y_0. Then, fix a line L_2 parallel to L_1 which separates x_0 from y_0, so that x_0 and y_0 lie in different components of $\mathbb{C}\setminus L_2$. Let U be the strip bounded by L_1 and L_2. Obviously, there exists an integer N such that P_n intersects both L_1 and L_2 for $n \geqslant N$. Without loss of generality, we may assume that every P_k intersects both L_1 and L_2.

It follows that for all $k \geq 1$ the intersection $\overline{U} \cap P_k = V \cap \overline{U} \cap P_k$ has a component Q_k intersecting both L_1 and L_2. Otherwise, by part (1) of Lemma 3.3 there is a separation $\overline{U} \cap P_k = A_1 \cup A_2$, where A_i is the union of all the components of $\overline{U} \cap P_k$ intersecting L_i. Let U_1, U_2 be the two components of $\mathbb{C}\setminus\overline{U}$, with $L_i = \partial U_i$. Then $P_k = [A_1 \cup (U_1 \cap P_k)] \ \cup \ [B_1 \cup (U_2 \cap P_k)]$ is a separation. This contradicts connectedness of P_k

Therefore, by part (3) of Lemma 3.3, our proof will be completed if only we can show that for all but two integers $k \geq 1$ the continuum Q_k is also a component of $\overline{U} \cap K$. To this end, for all $k \geq 1$ we may fix two points $a_k \in (L_1 \cap Q_k)$ and $b_k \in (L_2 \cap Q_k)$. Then, we fix a line L perpendicular to L_1 and disjoint from K. See Figure 4. Now, we call a continuum Q_k *a nearest*

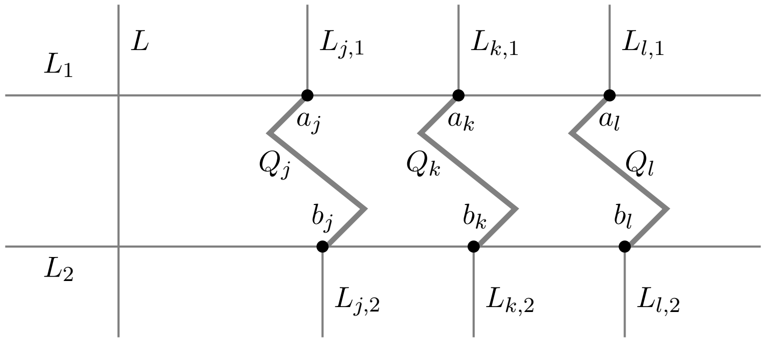

Figure 4: Relative locations of a_k, b_k and Q_j, Q_k, Q_l.

component (respectively *a furthest component*) if

$$\text{dist}\,(a_k, L) < \max\{\text{dist}\,(a_j, L) : j \neq k\} \quad (\text{dist}\,(a_k, L) > \max\{\text{dist}\,(a_j, L) : j \neq k\})\,.$$

Clearly, there exist at most one nearest component and at most one furthest component. We claim that all the other Q_k is also a component of $\overline{U} \cap K$. Actually, if U_i denotes the component of $\mathbb{C}\setminus\overline{U}$ with $\partial U_i = L_i$ we can choose for all $k \geq 1$ two rays $L_{k,1} \subset \overline{U_1}$ and $L_{k,2} \subset \overline{U_2}$ parallel to L such that $a_k \in L_{k,1}$ and $b_k \in L_{k,2}$. The above Figure 4 gives a simplified depiction for relative locations of L, L_i and a_j, a_k, a_l.

If Q_k is neither nearest nor furthest there exist two components Q_j, Q_l with $\text{dist}\,(a_j, L) < \text{dist}\,(a_k, L) < \text{dist}\,(a_l, L)$. In this case, we can use an appropriate brick wall tiling of U and the Jordan curve theorem to infer that $Q_k \setminus (L_1 \cup L_2)$ is contained in a bounded component W of

$\mathbb{C} \setminus M$, where $M = \overline{a_j a_l} \cup \overline{b_j b_l} \cup Q_j \cup Q_l$ is a continuum which lies entirely in V. Therefore, $(W \cup M) \cap K$ is a subset of $V \cap K$; and we are able to choose a separation $\overline{W} \cap K = A \cup B$ with $Q_k \subset A$ and $(Q_j \cup Q_l) \cap A = \emptyset$. From this we can infer that Q_k is also a component of A, which is disjoint from $B_1 := (\overline{U} \setminus W) \cap K$. That is to say, the intersection $\overline{U} \cap K$ is divided into two disjoint compact subsets, A and $B \cup B_1$. Combining this with the fact that Q_k is a component of A, we already verify that Q_k is also a component of $\overline{U} \cap K$. □

Finally, we prove Theorem 3.

Proof of Theorem 3. Suppose that (1) and (2) hold and assume that K does not satisfy the Schönflies condition. Then there exists a region U bounded by two parallel lines L_1 and L_2 such that $\overline{U} \cap K$ has infinitely many components $\{N_k\}$ which intersect both L_1 and L_2. Due to (2), there exists infinitely many $\{N_{k_i}\}$ in one component of K which is denoted by P, and these N_{k_i} are also in different components of $\overline{U} \cap P$. That is to say, P does not satisfy the Schönflies condition. By Theorem 2, we reach a contradiction to the local connectedness of P. This verifies the "if" part.

To prove the "only if" part we assume that K satisfies the Schönflies condition and verify conditions (1) and (2) as follows.

Given any component P of K and the region U bounded by any two parallel lines L_1 and L_2, it is routine to check that every component of $\overline{U} \cap P$ is also a component of $\overline{U} \cap K$. Thus $\overline{U} \cap P$ has finitely many components intersecting both L_1 and L_2. By Theorem 2, P is locally connected.

If (2) is not true, so that there exist an infinite sequence of sub-continua $\{N_k\}$ lying in distinct components of K, denoted Q_k, such that their diameters $\delta(N_k)$ are greater than a positive constant C, then we can choose a subsequence $\{N_{k_n}\}$ converging to a continuum $N_\infty \subset X$ under Hausdorff distance. Clearly, the diameter $\delta(N_\infty) \geq C$. So we can choose two points $x, y \in N_\infty$ with $|x - y| = C$ and two parallel lines L_1, L_2 perpendicular to the line crossing x, y and intersecting the interior of the segment $\overline{xy}$. Now we can see that all but finitely many $N_k \subset Q_k$ must intersect L_1 and L_2 at the same time. This implies that for all but finitely many integers $k \geq 1$, there exists a component P_k of $\overline{U} \cap Q_k$ intersecting both L_1 and L_2. Here U is the region bounded by L_1, L_2. For those integers k, the continuum P_k is also a component of $\overline{U} \cap K$, which contradicts the assumption that K satisfies the Schönflies condition. □

5 The Quotient $\mathcal{D}_K$ is a Peano Space

Throughout this section, we fix a compact set K in the plane and denote by $\sim$ the Schönflies equivalence relation given in Definition 4; moreover, the decomposition $\mathcal{D}_K$ of K is made up of the equivalence classes $[x] = \{y \in K : x \sim y\}$, for $x \in K$. Then $\mathcal{D}_K$ is an upper semi-continuous decomposition. In the following proposition we further show that its elements are all connected, so that the natural projection $\pi : K \to \mathcal{D}_K$ is a monotone map.

Proposition 5.1. *Every class* $[x]$ *of* $\sim$ *is a continuum.*

Proof. Assume that there exist two disjoint simple closed curves $J_1 \ni x$ and $J_2 \ni y$ such that the region W bounded by J_1 and J_2 has the following property: the common part $\overline{W} \cap K$ has infinitely many components $P_1, P_2, \ldots$ that intersect both J_1 and J_2 and converge to a continuum P_∞ under Hausdorff distance. We only need to verify that for any point $z \in P_\infty$ and any small number $r > 0$ there is a point y' with $|z - y'| < r$ such that y' is related to x under the relation R_K given before Definition 4. This then implies the inclusion $P_\infty \subset [x]$. By flexibility of x and y and by minimality of $\sim$ we will obtain that every class of $\sim$ is connected.

Considering K and W as subsets of the sphere and applying a Möbius transformation, if necessary, we may assume with no loss of generality that W is an open annulus. Then we may consider it as the standard annulus $\{1 < |z| < 2\}$. Let W be covered with a polar brick wall tiling, as indicated in the left part of Figure 5. Fix a separation $\overline{W} \cap K = A \cup B$ with $P_1 \subset A$

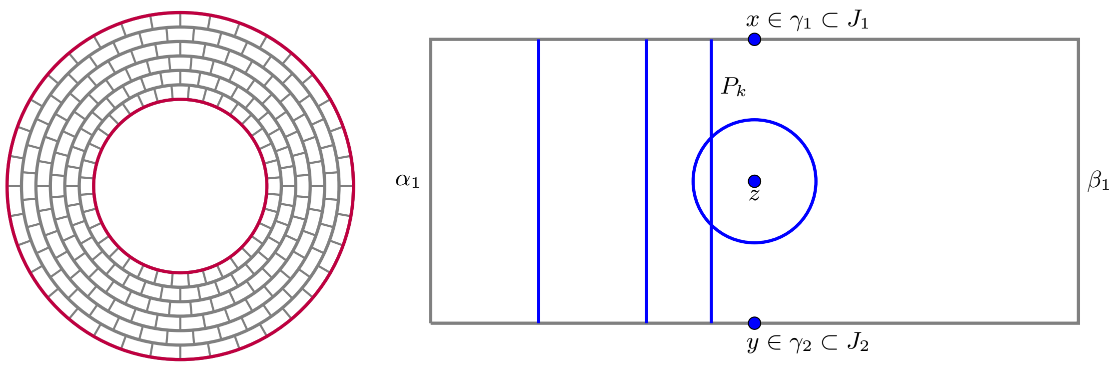

Figure 5: Polar Brick Wall Tiling and relative locations of P_k and $z \in W$.

and $P_\infty \subset B$. Assume that every tile in the polar brick wall tiling is of diameter less than $\frac{1}{3}\text{dist}(A, B)$, where $\text{dist}(A, B) := \inf\{|a - b| : a \in A, b \in B\}$ is the distance between A and B. Let A^* be the union of all the tiles intersecting A, and B^* the union of those intersecting B. Then each of A^* and B^* is a finite disjoint union of Peano continua. By the construction of polar brick wall tiling, each of those Peano continua has no cut point.

Let M be the component of A^* that contains P_1. By Torhorst Theorem [13, p.512, §61, II, Theorem 4], the unbounded component of $\mathbb{C} \setminus M$ is bounded by a simple closed curve Γ. Moreover, the curve Γ has exactly two arcs α_1, β_1 each of which connects a point on γ_1 to a point on γ_2. Then it is clear that $W \setminus (\alpha_1 \cup \beta_1)$ is the union of two regions, each of which is bounded by a simple closed curve. One of these two regions contains P_1 and the other contains P_∞. The latter is denoted D'_1 and depicted in the right part of Figure 5. Clearly, the boundary of D'_1 is the simple closed curve $J'_1 = \alpha_1 \cup \beta_1 \cup \gamma_1 \cup \gamma_2$.

Now, for any point $z \in (P_\infty \cap D'_1)$ and any number $r > 0$ that is smaller than $\mathrm{dist}(z, J'_1)$, we may fix a small open circular disk $D'_2 \subset D'_1$ centered at z with radius r and denote its boundary as J'_2. Recall that infinitely many P_k will intersect D'_2. For such a P_k the difference $P_k \setminus D'_2$ has a component Q_k intersecting both J'_1, J'_2. Since those Q_k are also components $\overline{U'} \cap K$, where $U' = D'_1 \setminus \overline{D'_2}$, and since we may choose an appropriate subsequence of $\{Q_k\}$ that is convergent under Hausdorff distance, we already show that $(x, y') \in R_K$ for some $y' \in J'_2$, which obviously satisfies the inequality $|z - y'| < r$. □

Now we assume that $\mathcal{D}_K$ is equipped with a metric d, which is compatible with the quotient topology. To prove Theorem 5, we start from a special case when the compactum K is a continuum.

Theorem 5.2. *If K is a continuum then $\mathcal{D}_K$ is locally connected under quotient topology.*

Proof. If $\mathcal{D}_K$ is not locally connected at some point $\pi(x)$, where x is a point in K and $\pi(x) = [x]$, then there exists a closed ball B_ε in $\mathcal{D}_K$ centered at $\pi(x)$ with radius $\varepsilon > 0$ whose component containing the point $\pi(x)$ is not a neighborhood of $\pi(x)$. Let Q be the component of B_ε containing $\pi(x)$. Then we can find an infinite sequence $\pi(x_k)$ in $B_\varepsilon \setminus Q$ with $d(\pi(x_k), \pi(x)) \to 0$ such that for $k \neq l$ the components $Q_k \ni \pi(x_k)$ and $Q_l \ni \pi(x_l)$ are disjoint.

Fix a point $\pi(y_k)$ on $Q_k \cap \partial B_\varepsilon$ for all $k \geq 1$. By coming to an appropriate subsequence, if necessary, we may assume $x_k \to x_\infty$ and $y_k \to y_\infty$. Then, continuity of π guarantees that $\pi(x_\infty) = \lim\limits_{k\to\infty} \pi(x_k) = \pi(x)$ and that $\pi(y_\infty) = \lim\limits_{k\to\infty} \pi(y_k)$ belongs to the boundary of B_ε. Now, considered as subsets of K, we see that $\pi(x_\infty) = [x_\infty]$ and $\pi(y_\infty) = [y_\infty]$ are disjoint planar continua. In particular, we have $[x_\infty] \subset \pi^{-1}(B_\varepsilon^o)$ and $[y_\infty] \subset E := \pi^{-1}[(\mathcal{D}_K) \setminus B_\varepsilon^o] = K \setminus \pi^{-1}(B_\varepsilon^o)$, where B_ε^o denotes the interior of B_ε. Since $\pi : K \to \mathcal{D}_K$ is a monotone map, the pre-images $\pi^{-1}(Q), \pi^{-1}(Q_k)$ are disjoint sub-continua of K. Moreover, there exist two disjoint open sets V_1 and V_2 satisfying:

$$[x_\infty] \subset V_1 \subset \pi^{-1}(B_\varepsilon^o), \qquad [y_\infty] \subset E \subset V_2, \qquad \pi(V_1) \cap \pi(V_2) = \emptyset. \tag{1}$$

By compactness of K we may assume that K is contained in a strip $\{x+\mathbf{i}y : x\in\mathbb{R}, 0\le y\le 1\}$, which will be covered by rectangles of the form

$$R_{k,j}^n := \left\{x+\mathbf{i}y:\ \frac{2k}{2^n}+\frac{1-(-1)^j}{2^{n+1}}\le x\le \frac{2k+2}{2^n}+\frac{1-(-1)^j}{2^{n+1}},\ \ \frac{j}{2^n}\le y\le\frac{j+1}{2^n}\right\}. \tag{2}$$

with $n\ge 1$, $k\in\mathbb{Z}$ and $0\le j\le 2^n-1$. Then $\left\{R_{k,j}^n\right\}$ is a tiling of $\mathbb{R}\times[0,1]$; every $R_{k,j}^n$ is called a tile and every two tiles either are disjoint or intersect at a non-degenerate segment. Let T_n be the union of all the tiles $R_{k,j}^n$ that intersects E, and S_n the union of those that intersect $[x_\infty]$. Then each of T_n, S_n is the union of finitely many rectangles and S_n is connected. Clearly, they have no cut points; moreover, the interior of T_n contains E and that of S_n contains $[x_\infty]$.

Choose a large integer $n\ge 1$ satisfying $S_n\subset V_1$ and $T_n\subset V_2$, so that $\pi(T_n)\cap\pi(S_n)=\emptyset$. Let W_1 be the component of $\mathbb{C}\setminus S_n$ containing $[y_\infty]$. Since S_n is locally connected and has no cut point, by Torhorst Theorem [21, p.126] we see that the boundary $J_1:=\partial W_1$ is a simple closed curve. Moreover, the difference $W_1\setminus T_n$ is an open set (possibly not connected) bounded by two or more disjoint simple closed curves. It has a unique component W^* whose boundary contains J_1 and all the other components of $W_1\setminus T_n$ have a positive distance to J_1. Since the equivalence class $[y_\infty]$ is a continuum lying in W_1, the boundary of W^* contains at least another simple closed curve J_2, which separates J_1 from $[y_\infty]$, and consists of finitely many disjoint simple closed curves, say $J_1, J_2, \ldots, J_m$. Since $J_1\subset S_n$ and $(J_2\cup J_3\cup\cdots\cup J_n)\subset T_n$, we have

$$\pi(J_1)\cap\pi(J_2\cup\cdots\cup J_m)=\emptyset. \tag{3}$$

The rest of our proof is to obtain a contradiction to this equation. To this end, we firstly recall that, for every large enough $k>1$, the pre-image $\pi^{-1}(Q_k)$ is a continuum intersecting both J_1 and J_2. Then it follows that $\overline{W^*}\cap\pi^{-1}(Q_k)$ has a component P_k intersecting both J_1 and some J_i with $2\le i\le m$. Let $2\le i_0\le m$ be an integer such that that $P_k\cap J_{i_0}\ne\emptyset$ for infinitely many k.

We may assume that $J_2\subset Int(J_1)$, by applying a Möbius transformation if necessary.

Since $\overline{W^*}\cap E=\emptyset$, we have $\left(\overline{W^*}\cap K\right)\subset\pi^{-1}(B_{\varepsilon-\xi})$ for a small $\xi\in(0,\varepsilon)$. Combing this with the fact that the continuum $\pi^{-1}(Q_k)$ is a component of $\pi^{-1}(B_\varepsilon)$, we can infer that P_k is also a component of $\overline{W^*}\cap K$. Choose a subsequence of $\{P_k\}$ that converges to a limit continuum P_∞ under Hausdorff distance. Then P_∞ intersects J_1 and J_{i_0} at the same time. By Lemma 3.5, we have $(z_1,z_2)\in R_K$ for all $z_1\in(J_1\cap P_\infty)$ and $z_2\in(J_{i_0}\cap P_\infty)\subset(J_2\cup\cdots\cup J_m)$. This is impossible by Equation (3). □

Then, we discuss the case when K is a disconnected compactum.

Proof for Theorem 5. By Theorem 5.2, we just need to show that for any $\varepsilon > 0$ there are at most finitely many components of $\mathcal{D}_K$ which are of diameter greater than ε.

Otherwise, there is an infinite sequence $\{Q_j : j\}$ of components whose diameters are greater than a constant $\varepsilon_0 > 0$. By monotonicity of the natural projection $\pi : K \to \mathcal{D}_K$, the preimages $P_j := \pi^{-1}(Q_j)$ are components of K. By uniform continuity of π, the diameters of P_i are greater than a constant $t_0 > 0$. By going to a subsequence, if necessary,we may assume that the sequence $\{P_j\}$ converges to a limit continuum P_∞ under Hausdorff distance. Then, continuity of π ensures that $Q_j \to \pi(P_\infty)$, which is a continuum of diameter $\geq \varepsilon_0$. So, we can fix two points $x_1, x_2 \in P_\infty$ with $d(\pi(x_1), \pi(x_2)) \geq \varepsilon_0$. Clearly, the two equivalence classes $[x_1], [x_2]$ are disjoint continua in the plane.

Since $\mathcal{D}_K = \{[z] : z \in K\}$ is an upper semi-continuous decomposition, we may fix two open sets $U_1 \supset [x_1]$ and $U_2 \supset [x_2]$ with disjoint closures such that $[z] \cap U_2 = \emptyset$ for all $z \in U_1$.

Let J_i be the circle centered at x_i with an arbitrary radius $r > 0$ such that $J_i \subset U_i$. Let W be the component of $\mathbb{C} \setminus (J_1 \cup J_2)$ with $\partial W = J_1 \cup J_2$. The containment $\{x_1, x_2\} \subset P_\infty$ implies that all but finitely many P_j intersect both J_1 and J_2. Recall that every P_j is a component of K. For each of those P_j intersecting both J_1 and J_2, the intersection $\overline{W} \cap K$ has a component M_j, which intersects J_1 and J_2 both. Those M_j are each a component of a component of $\overline{W} \cap P_j$. Therefore, a subsequence of $\{M_j\}$ must converge to a limit continuum M_∞, which intersects both J_1 and J_2. Fixing any point $z \in M_\infty \cap J_1$, we have $z \in U_1$ and $([z] \cap U_2) \supset ([z] \cap J_2) \neq \emptyset$. This contradicts the fact mentioned in the previous paragraph. □

6 $\mathcal{D}_K$ is a Core Decomposition

This section has a single aim, to prove Theorem 6. And we **only need to show** that $f(x_1) = f(x_2)$ for any $x_1, x_2 \in K$ with $(x_1, x_2) \in R_K$, since $\sim$ is the smallest closed equivalence on K containing R_K and since $\{f^{-1}(y) : y \in Y\}$ is also a monotone decomposition of K.

Proof for Theorem 6. By Definition 4, we may fix two disjoint simple closed curves $J_i \ni x_i$ such that $\overline{W} \cap K$ has infinitely many components P_k, each of which intersects both J_1 and J_2, such that the sequence $\{P_k\}$ converges to a continuum $P_\infty \supset \{x_1, x_2\}$ under Hausdorff distance. Here W is the only component of $\mathbb{C} \setminus (J_1 \cup J_2)$ with $\partial W = J_1 \cup J_2$.

By applying an appropriate Möbius transformation, if necessary, we may assume with no loss of generality that J_1 separates J_2 from infinity. By Schönflies Theorem (see [15, p.71, Theorem

3] or [15, p.72, Theorem 4]), we may consider $\overline{W}$ to be the annulus $A = \{z : 1 \leq |z| \leq 2\}$.

In the rest of this section, we denote by ρ the metric on Y and assume on the contrary that $f(x_1) \neq f(x_2)$. Then we could find a positive number $\varepsilon_0 < \frac{1}{2}\rho(f(x_1), f(x_2))$ such that the two disks $B_1, B_2 \subset Y$ centered at x_1, x_2 with radius ε_0 have disjoint closures, *i.e.* $\overline{B_1} \cap \overline{B_2} = \emptyset$. Let $U_i = f^{-1}(B_i)$ for $i = 1, 2$. Then

$$\overline{U_1} \cap \overline{U_2} = \emptyset = f\left(\overline{U_1}\right) \cap f\left(\overline{U_2}\right). \tag{4}$$

Since $\lim_{k\to\infty} f(P_k) = f(P_\infty)$ under Hausdorff distance, all but finitely many $f(P_k)$ are of diameter greater than $2\varepsilon_0$. Since Y is a Peano space, all but finitely many $f(P_k)$ must be entirely contained in a single component of Y, denoted Y_0, which also contains $f(P_\infty)$. With no loss of generality we may assume that every $f(P_k)$ is entirely contained in Y_0.

To complete our proof, we will induce a contradiction to local connectedness of Y_0, by showing that Y_0 is not locally connected at some point on $f(P_\infty)$. More precisely, let $K_0 = K \setminus (U_1 \cup U_2)$; then we will find a point $y^\# \in f(P_\infty) \subset Y_0$ such that $f(K_0)$ is a neighborhood of $y^\#$ and that the component Q_0 of $f(K_0)$ containing $y^\#$ is not a neighborhood of $y^\#$. This clearly indicates that the component of $f(K_0) \cap Y_0$ containing $y^\#$ is not a neighborhood of $y^\#$.

For every $k \geq 1$, we can choose a separation $A \cap K = E_k \cup F_k$ such that $P_k \subset E_k$, $P_\infty \subset F_k$, and $P_j \subset F_k$ for $j \neq k$. Cover the annulus A by a polar brick wall tiling $\mathcal{T}_k$, whose tiles are all of diameter strictly smaller than $\frac{1}{2}\text{dist}(E_k, F_k)$. Let P_k^* be the component of

$$\bigcup \{T \in \mathcal{T}_k : T \cap E_k \neq \emptyset\}$$

containing P_k. Then the unbounded component W_k of $\mathbb{C} \setminus P_k^*$ contains P_∞ and every P_j with $j \neq k$. Since P_k^* is a continuum with no cut point, by Torhorst Theorem [21, p.126], the boundary of W_k is a simple closed curve Γ_k. Moreover, Γ_k contains exactly two sub-arcs α_k, β_k each of which intersects both $C_1 := \{|z| = 1\}$ and $C_2 := \{|z| = 2\}$ and is otherwise contained in the open annulus A^o. Let D_k be the component of $\mathbb{C} \setminus (C_1 \cup C_2 \cup \alpha_k \cup \beta_k)$ that contains P_k. Then ∂D_k is a simple closed curve for all $k \geq 1$.

By appropriately choosing the sizes of tiles in $\mathcal{T}_k$, we may further assume that the topological disks D_k are pairwise disjoint. Then P_∞ is contained in the difference $A \setminus D_1$, whose closure is also a topological disk, denoted as D_∞. Now consider D_∞ as the unit square $[0, 1]^2$.

Recall that $D_\infty \cap K$ has infinitely many components P_k that intersects the top and the bottom of $[0, 1]^2$ at the same time. Therefore, the limit P_∞ of a subsequence of $\{P_k\}$ intersects each of the two segments $[0, 1] \times \{0\}$ and $[0, 1] \times \{1\}$. Let P^* be the component of $D_\infty \cap K$ that contains P_∞.

Let the two components of $[0,1]^2 \setminus P^*$ containing $\{0\} \times [0,1]$ and $\{1\} \times [0,1]$ be respectively denoted as U_L and U_R. Then, one of U_L and U_R contains infinitely many components of $D_\infty \cap K$. Assume that U_L contains an infinite subsequence of $\{P_k\}$ and rename this subsequence as $\{Q_k\}$. Then all but finitely many Q_k intersect the two segments $[0,1] \times \{0\}$ and $[0,1] \times \{1\}$ both.

Now, fix two disks $B(x_i, r)$ centered at x_i with radius $r > 0$ satisfying $B(x_i, r) \subset U_i$. Since $Q_k \to P_\infty \subset P^*$ under Hausdorff distance, we may find two points $x_{k,i}$ in $(0,1)^2 \cap Q_k$ for a large $k \geq 1$ such that both $|x_{k,1} - x_1|$ and $|x_{k,2} - x_2|$ are smaller than $\min\left\{\frac{1}{2}, r\right\}$. Then, the segments

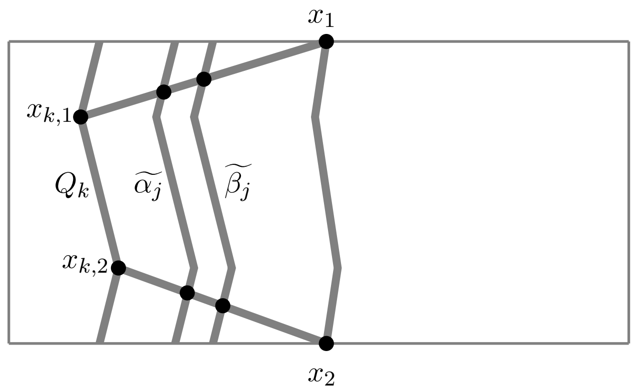

Figure 6: Relative locations of the component P_k, the arcs $\widetilde{\alpha_j}, \widetilde{\beta_j}$ and the points a_j, b_j, c_j, d_j.

$\gamma_{k,i} := \overline{x_{k,i} x_i}$ for $i = 1, 2$ are disjoint and both $\gamma_{k,1}$ and $\gamma_{k,2}$ intersect all but finitely many of the topological disks D_j that are constructed by using the polar brick wall tiling $\mathcal{T}_j$. In particular, each of $\gamma_{k,1}, \gamma_{k,2}$ intersects both α_j and β_j for all j greater than some integer $N \geq k$. For each of those $j > N$, let a_j be the last point of $\gamma_{k,1}$ that leaves α_j and b_j the first point of $\gamma_{k,1}$ after a_j that lies on β_j; let c_j be the last point of $\gamma_{k,2}$ that leaves α_j and d_j the first point of $\gamma_{k,2}$ after c_j that lies on β_j. Then, the segments $\overline{a_j b_j}, \overline{c_j d_j}$, the arc $\widetilde{\alpha_j} \subset \alpha_j$ connecting a_j to b_j and the arc $\widetilde{\beta_j} \subset \beta_j$ connecting c_j to d_j form a simple closed curve, denoted γ_j. Since the disks D_j are disjoint, so are the disks $\Delta_j \subset D_j$ that are bounded by γ_j. For all the integers $j > N$, we have two facts that will be used before we end our proof.

1. For all $j \geq 1$, the intersection $K \cap \left(\overline{a_j b_j} \cup \overline{c_j d_j}\right) = K \cap (\partial \Delta_j) = K \cap \gamma_j$ is contained in $B(x_1, r) \cup B(x_2, r)$, which is a subset of $U_1 \cup U_2 = f^{-1}(B_1) \cup f^{-1}(B_2)$.

2. The intersection $\overline{\Delta_j} \cap K$ has a component that intersects the segments $\overline{a_j b_j}$ and $\overline{c_j d_j}$ at the same time. Denote this component by M_j.

Since $\lim_j \overline{a_j b_j} = x_1$ and $\lim_j \overline{c_j d_j} = x_2$, for every $j > N$ we may choose a point $z_j \in M_j$ such that the distance from $f(z_j)$ to B_1 equals that from $f(z_j)$ to B_2. Choose a convergent subsequence of $\{z_j\}$, denoted as $\{v_n\}$, that converges to a point $v_\infty \in P_\infty$. Then the distance from $y^\# = f(v_\infty)$ to B_1 equals that from $y^\#$ to B_2. This implies that $y^\#$ lies in the interior of $f(K_0) = Y \setminus (B_1 \cup B_2)$, where

$$K_0 = K \setminus (U_1 \cup U_2) = K \setminus \left(f^{-1}(B_1) \cup f^{-1}(B_2)\right).$$

Note that $\{f(v_n) : n\}$ is an infinite sequence of points in $f(K_0)$ converging to $y^\#$ such that the components of $f(K_0)$ containing $f(v_n)$ are pairwise disjoint. This implies that the component Q_0 of $f(K_0)$ containing $y^\#$ is not a neighborhood of $y^\#$ and ends the proof. □

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