Interval exchange transformations

Teichmüller theory through the eyes of word combinatorics

Vincent Delecroix

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Lecture 1: Interval exchange maps

- Rauzy induction: a particular case of S-adic system (. . . to be continued in Lecture 2)
- coding of translation flows (and billiards)

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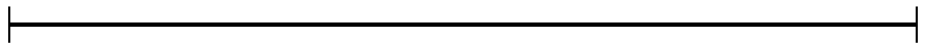

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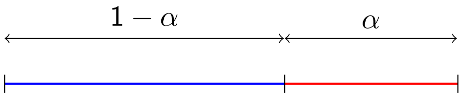

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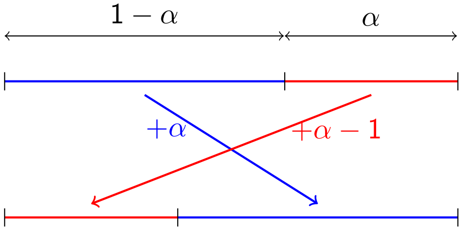

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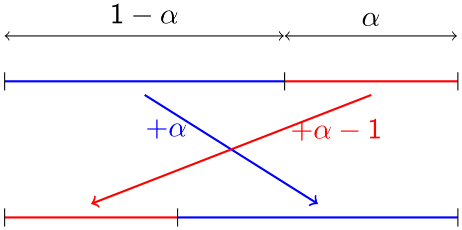

We aim to study the dynamics of such map.

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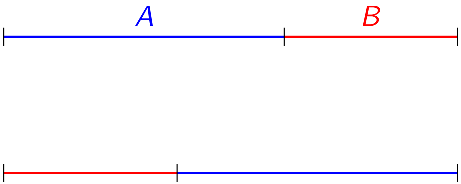
- **Warning:** T_α is *not* continuous on $[0, 1)$.

To consider the topological side, we consider a the associated coding $\widehat{T} : X_\alpha \to X_\alpha$ where $X_\alpha \subset \{A, B\}$ ($\widehat{T}$ is the shift map on sequences).

Coding



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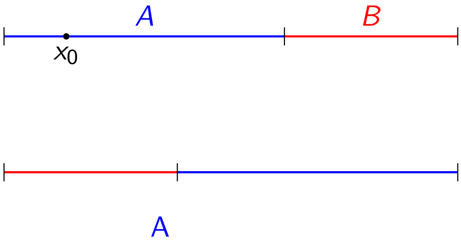

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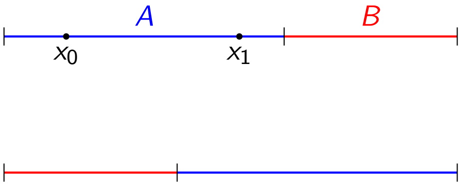
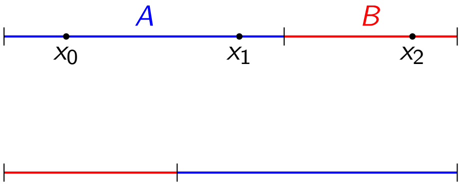


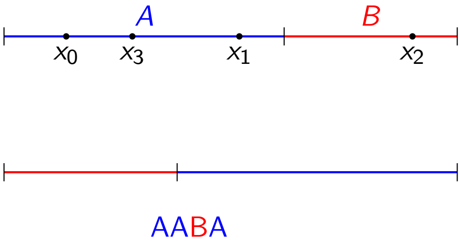
AA

Coding



AAB

Coding



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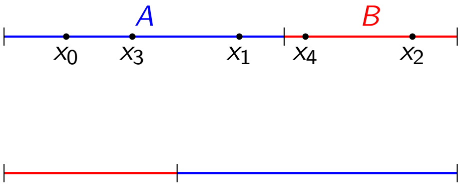

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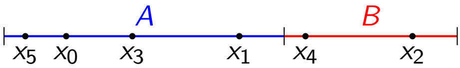

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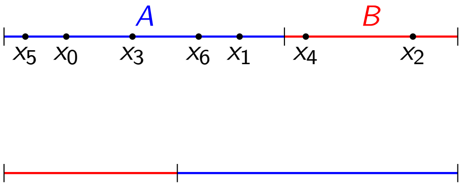

AABABAA

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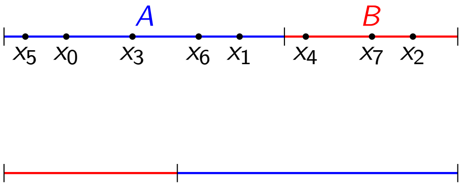

AABABAAB...

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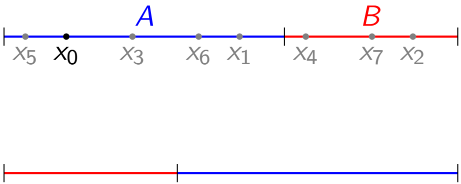
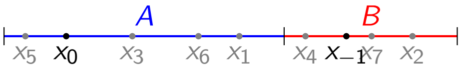


$.AABABAAB\ldots$

Coding



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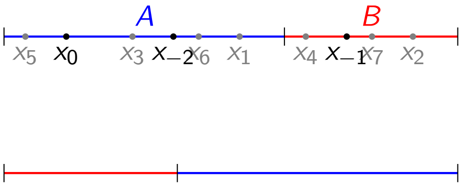

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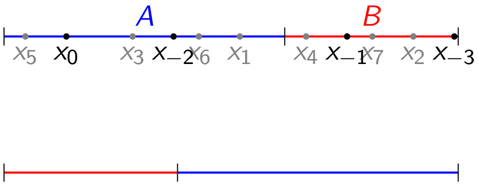

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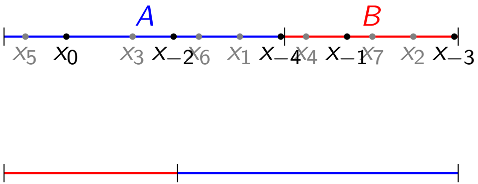

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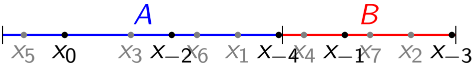

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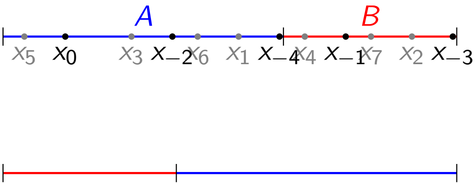

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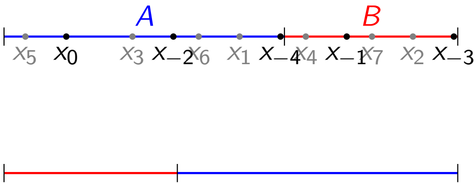

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From here two options to construct $X_\alpha \subset \{A, B\}^{\mathbb{Z}}$:

- take the closure of the set of codings of regular sequences,
- define the coding**s** of singular sequences ("Keane construction").

Coding

Theorem

If α is irrational, there is a unique continuous surjective map $p : X_\alpha \to [0,1]$ so that the coding of $p(w)$ is w. All points have exactly one preimage except the singular orbits that have two.

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The singular orbits have codings $\omega_- AB\omega_+$ and $\omega_- BA\omega_+$.

Dynamical results

Theorem

Let α be irrational, and X_α be the Sturmian shift associated to the rotation T_α. Then:

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- *the shift X_α is* **minimal** *(all orbits are dense);*
- *(Hecke (1922), Ostrowski (1922)) any clopen $Y \subset X_\alpha$ has bounded remainder: there exists μ_Y and C_Y so that*

$$\forall x \in X_\alpha, \forall n \geq 0, \quad \left| \sum_{k=0}^{n} \Big(\chi_Y(T_\alpha^k x) - \mu_Y \Big) \right| \leq C_Y.$$

In particular, the shift X_α is **uniquely ergodic**.

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remark: for the clopens $Y = [A]$ or $Y = [B]$ we can pick $C_Y = 1$ (1-balancedness).

Rauzy induction, continued fractions

For a pair of positive real numbers $\lambda = (\lambda_A, \lambda_B)$ we consider the map $T_\lambda : [0, |\lambda|] \to [0, |\lambda|]$ given by

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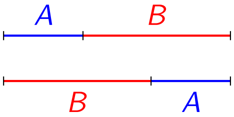
The *Rauzy induction* is the procedure which associates to the map T_λ the induced map on $[0, \max(\lambda_A, \lambda_B)]$.

Rauzy induction and continued fractions

top induction

case $\lambda_B > \lambda_A$



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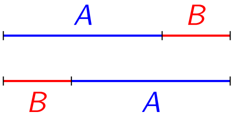

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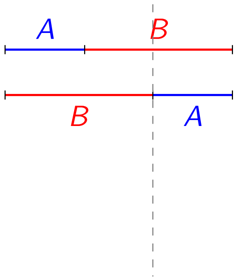

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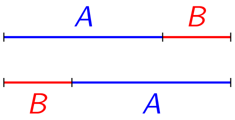

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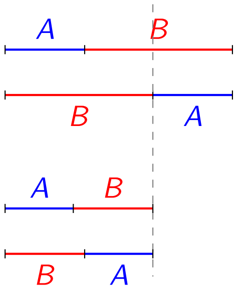

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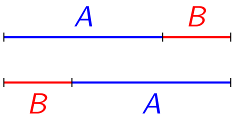

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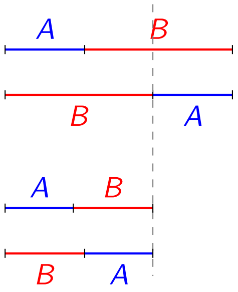

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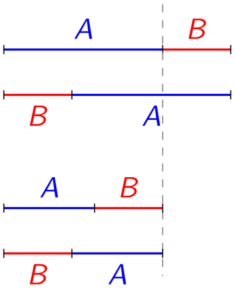

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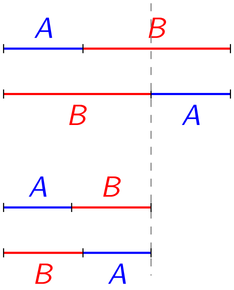

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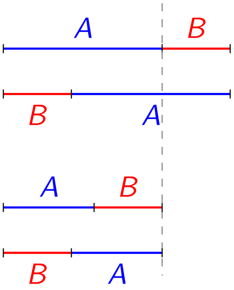

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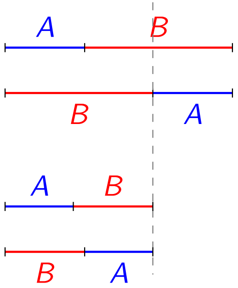

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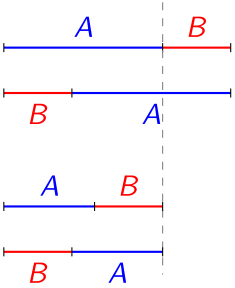

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Rauzy induction, continued fractions

Theorem

The Rauzy induction (or Farey map) associates to a rotation T with lengths (λ_A, λ_B) the new rotation T' with either lengths $(\lambda_A, \lambda_B - \lambda_A)$ ("top type") or $(\lambda_A - \lambda_B, \lambda_B)$ ("bot type").

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The codings for T are recovered from the coding of T' by applying one of the substitution

$$\sigma^{top} : \begin{cases} A \mapsto AB \\ B \mapsto B \end{cases} \quad \text{or} \quad \sigma^{bot} \begin{cases} A \mapsto A \\ B \mapsto AB \end{cases} .$$

Rauzy induction, continued fractions

Let

$$A(\lambda) = \begin{cases} A^{top} & \text{if } \lambda_A < \lambda_B, \\ A^{bot} & \text{if } \lambda_A > \lambda_B. \end{cases}$$

where

$$A^{top} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad A^{bot} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

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$$A_n(\lambda) = A(\lambda)A(R\lambda)\ldots A(R^{n-1}\lambda).$$

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Note that A (and Rauzy induction R) commutes with scalar multiplication $A(e^s\lambda) = e^s A(\lambda)$.

Rauzy induction, continued fractions

Because

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we can write

$$\frac{\lambda_B}{\lambda_A} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\dots}}}.$$

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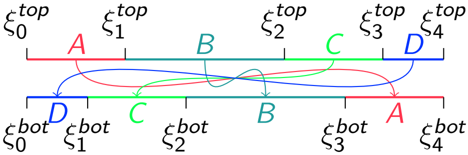
$$\frac{\lambda_B}{\lambda_A} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\dots}}}.$$

This is called the *continued fraction* of λ_B/λ_A.

Interval exchange transformations

An *interval exchange transformation* T is a piecewise translation of an interval



$$T : I \backslash \{\xi_1^{top}, \ldots, \xi_{d-1}^{top}\} \to I \backslash \{\xi_1^{bot}, \ldots, \xi_{d-1}^{bot}\}.$$

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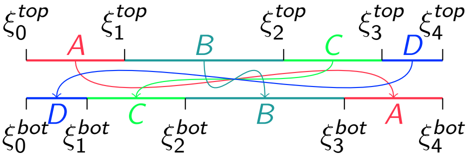

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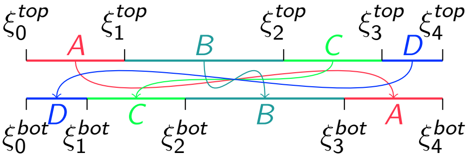
The above interval exchange transformation can be defined from:

- a "permutation" $\pi = \begin{pmatrix} \pi^{top} \\ \pi^{bot} \end{pmatrix} = \begin{pmatrix} A & B & C & D \\ D & C & B & A \end{pmatrix}$,
- a length vector $\lambda = (\lambda_A, \lambda_B, \lambda_C, \lambda_D)$.

Interval exchange transformations

An *interval exchange transformation* T is a piecewise translation of an interval



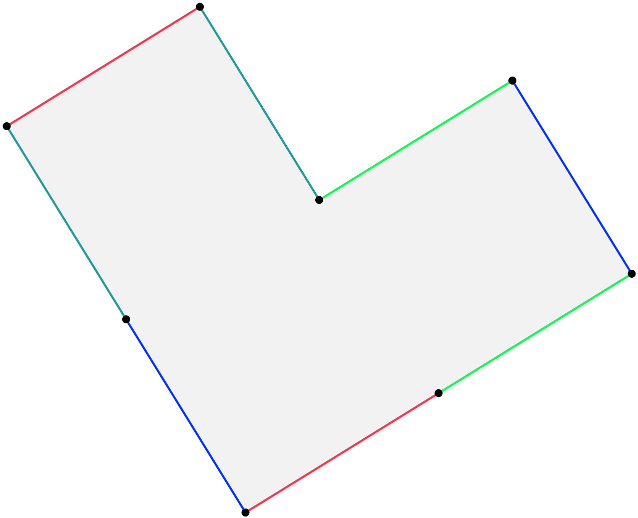
$T : I \backslash \{\xi_1^{top}, \ldots, \xi_{d-1}^{top}\} \to I \backslash \{\xi_1^{bot}, \ldots, \xi_{d-1}^{bot}\}$.

The above interval exchange transformation can be defined from:

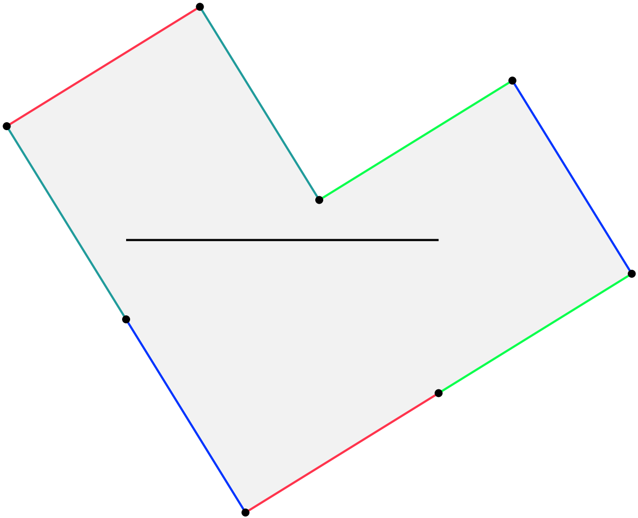
- a "permutation" $\pi = \begin{pmatrix} \pi^{top} \\ \pi^{bot} \end{pmatrix} = \begin{pmatrix} A & B & C & D \\ D & C & B & A \end{pmatrix}$,
- a length vector $\lambda = (\lambda_A, \lambda_B, \lambda_C, \lambda_D)$.

We call $\{\xi_1^{top}, \ldots, \xi_{d-1}^{top}\}$ (respectively $\{\xi_1^{bot}, \ldots, \xi_{d-1}^{bot}\}$) the *top singularities* (resp. *bot singularities*) of T.

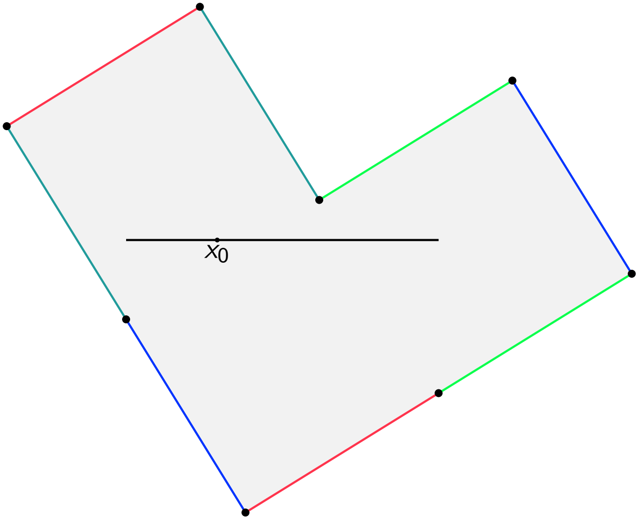
Translation surfaces



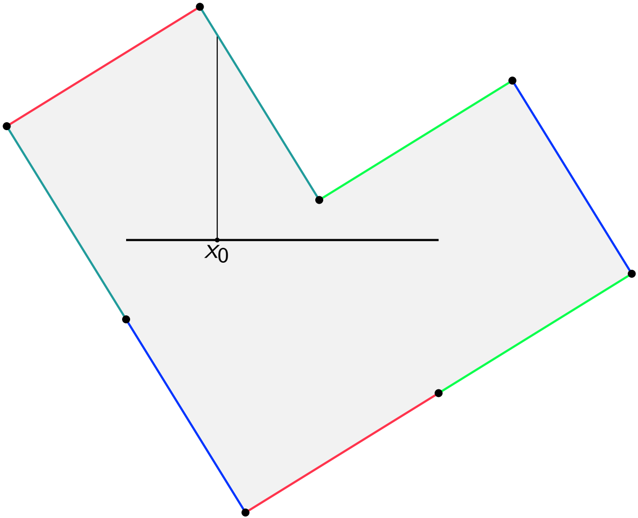
Translation surfaces



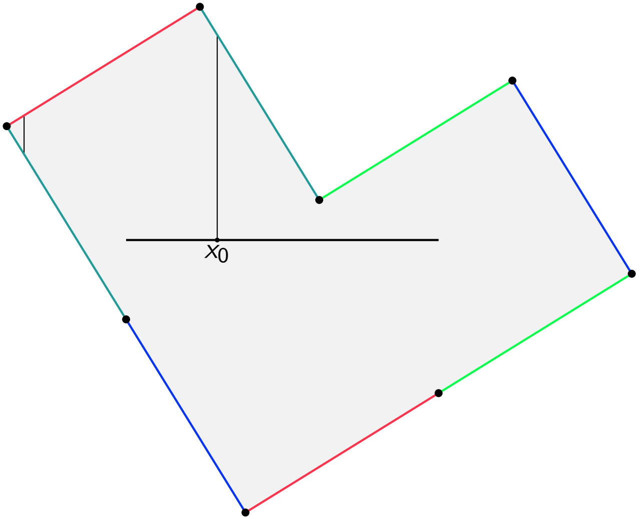
Translation surfaces



Translation surfaces



Translation surfaces



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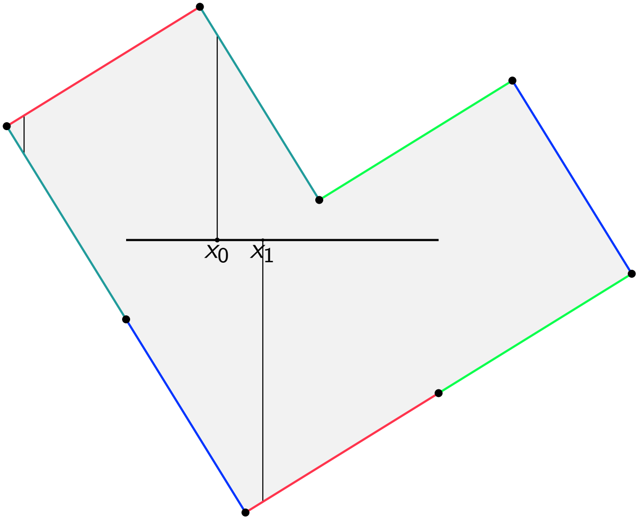
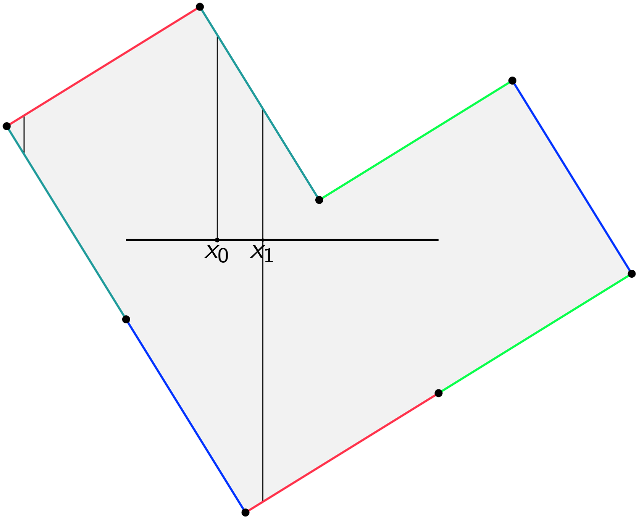
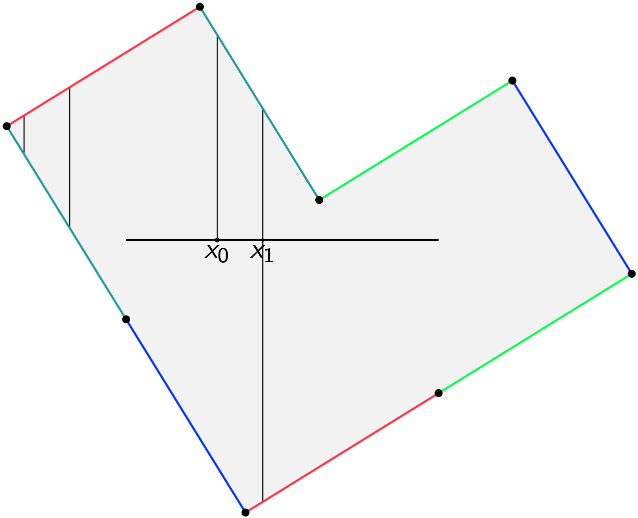


Translation surfaces



Translation surfaces



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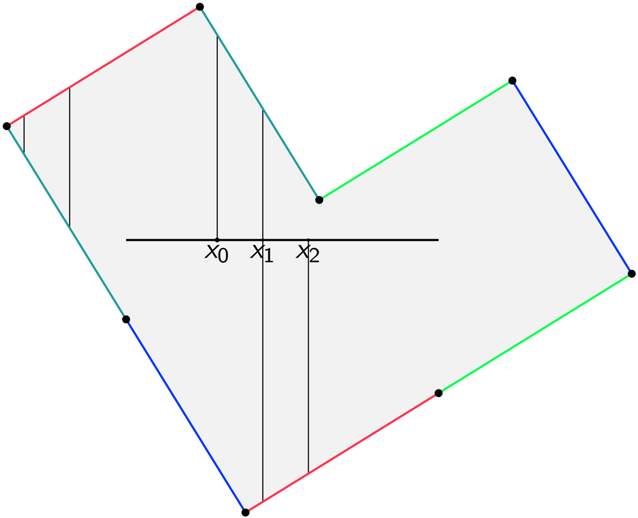
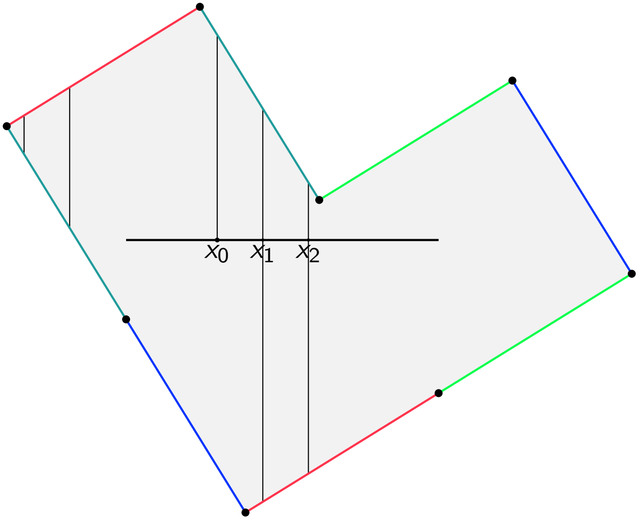
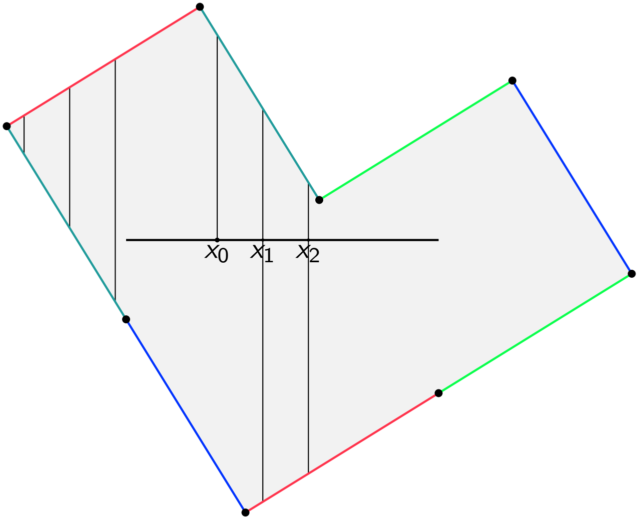


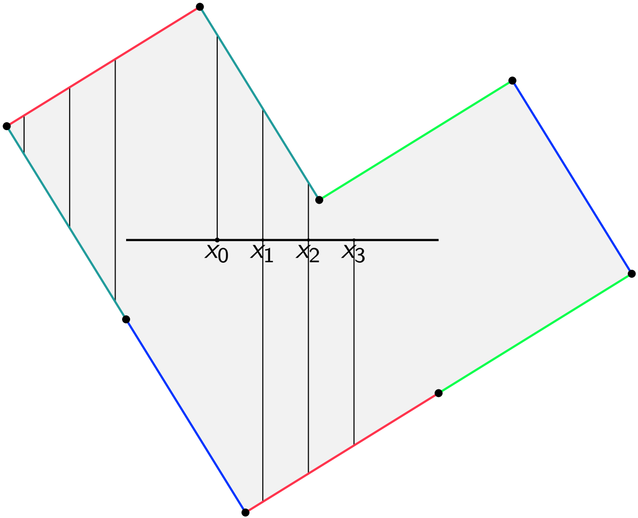
Translation surfaces



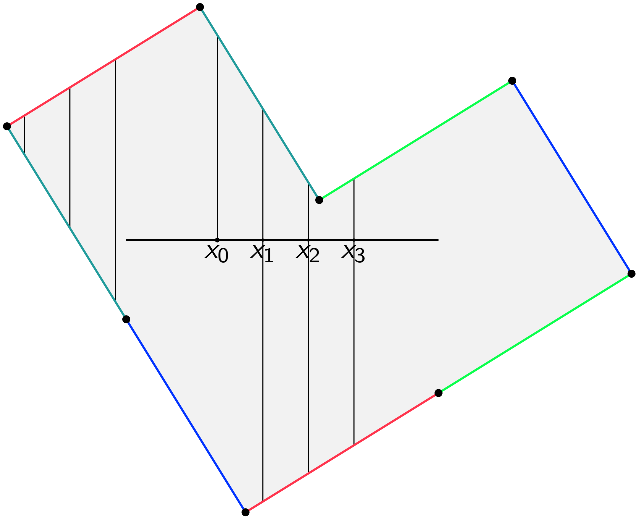
Translation surfaces



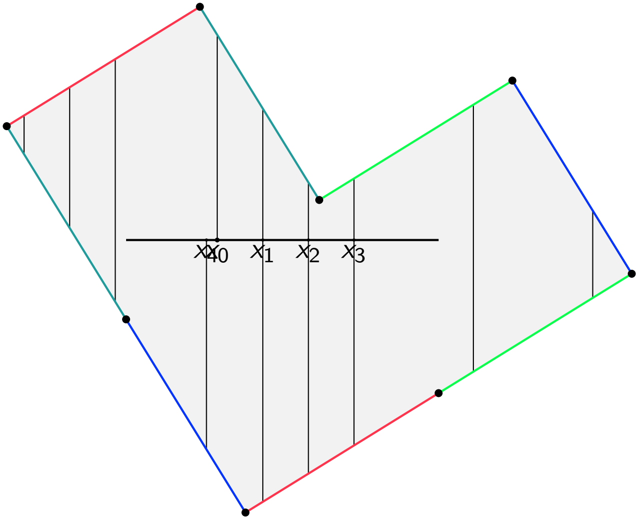
Translation surfaces



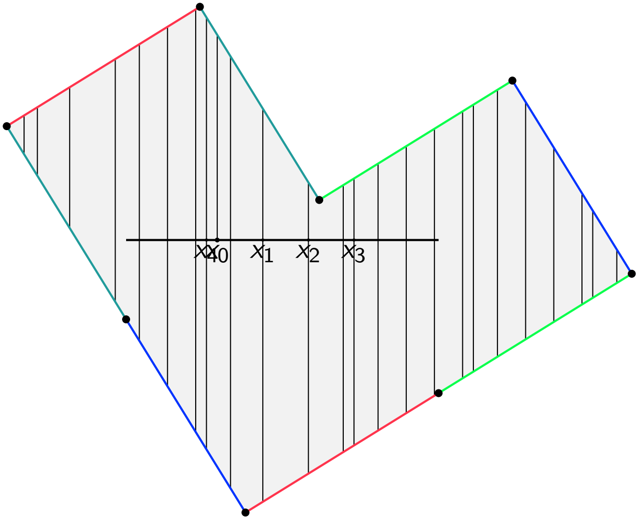
Translation surfaces



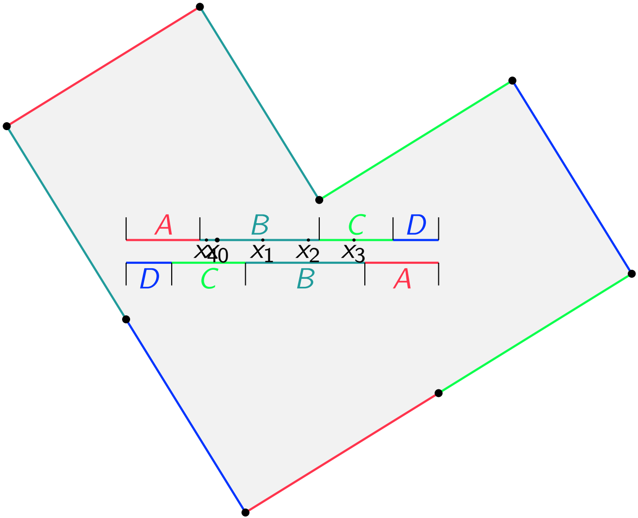
Translation surfaces



Translation surfaces



Translation surfaces



From translation surfaces to interval exchanges

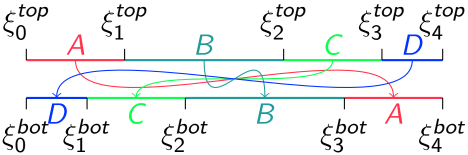
Theorem

Let S be a translation surface with s conical singularities. Let $I \subset S$ be an horizontal segment so that

- *each leaf of the vertical flow meets I,*
- *both endpoints of I have the property that either in the past or the future, they bump into a singularity of the surface before coming back to the interval.*

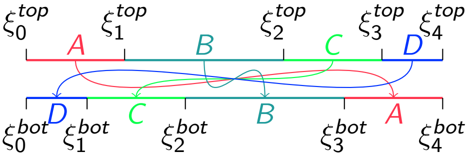
Then the Poincaré map of the vertical flow on I is an interval exchange map on $2g + s - 1$ intervals.

Coding



As we did for rotations, given the interval exchange transformation T above, we could code orbits in $\{A, B, C, D\}^{\mathbb{Z}}$ (except the singular ones). We obtain a shift $\widehat{T} : X_{\pi,\lambda} \to X_{\pi,\lambda}$ and a factor map $p : X_{\pi,\lambda} \to I$.

Coding



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Each regular orbit of the iet $T_{\pi,\lambda}$ has one preimage in $X_{\pi,\lambda}$ except the singular ones that have two.